Portfolio Selection under Ambiguity in Volatility*

Prince Osei [1] and Frank Riedel [1]

[1]Center for Mathematical Economics, Bielefeld University, Bielefeld, Germany.

February 20, 2026

Abstract

We study optimal portfolio choice when the variance of asset returns is ambiguous. Building on the smooth model of ambiguity aversion by Klibanoff et al. (2005), we introduce a one-period framework in which returns follow a Variance–Gamma specification, obtained by mixing a normal distribution with a gamma prior on the variance. This structure captures empirically observed excess kurtosis and allows us to derive closed-form solutions for optimal demand.

Our main results show that ambiguity about volatility leads to bounded portfolio positions, in sharp contrast to the unbounded exposures predicted by the classical CAPM when expected excess returns are large or when the mean variance tends to zero. We characterize the comparative statics of the optimal allocation with respect to risk aversion, ambiguity aversion, and the parameters of the prior distribution. For small mean excess returns, portfolio demand converges to the CAPM benchmark, indicating that ambiguity aversion affects higher-order terms only. The model provides a tractable link between robust portfolio choice and realistic, heavy-tailed return dynamics.

Keywords: Portfolio choice, Volatility uncertainty, Smooth ambiguity aversion, Variance–Gamma distribution.

JEL Classification: D81, G11, C46.

1 Introduction

Modern portfolio theory, since the seminal work of Markowitz (1952), has provided a quantitative framework for optimal investment under risk. The classical Capital Asset Pricing Model (CAPM) and its continuous-time extensions by Merton (1969, 1971)

*Email addresses: prince.osei@uni-bielefeld.de (P. Osei) and frank.riedel@uni-bielefeld.de (F. Riedel).
We thank Jean-Marc Tallon for fruitful discussions. Financial support by the German Research Foundation (DFG) [RTG 2865/1 – 492988838] is gratefully acknowledged.

assume that investors know the distribution of returns, in particular their mean and variance. In practice, however, both parameters are estimated with uncertainty, and empirical evidence suggests that ambiguity about volatility is especially pronounced in financial markets. Periods of turbulence, model misspecification, and structural change expose investors to higher-order uncertainty that standard mean–variance analysis cannot capture.

This paper develops a tractable model of portfolio selection when the *variance of returns is ambiguous*. We embed volatility uncertainty in a smooth model of ambiguity aversion à la Klibanoff et al. (2005) and show how it modifies classical portfolio demand. Specifically, we assume that asset returns follow a Variance–Gamma specification, obtained by mixing normal returns with a Gamma prior on their variance. Introduced by Madan & Seneta (1990), this widely used framework captures the semi-heavy tails of empirical return distributions and overcomes the limitations of normality. The resulting distribution is more peaked around the mean and exhibits excess kurtosis while retaining finite moments, offering a convenient and flexible way to model non-normal daily returns.

We are able to obtain an explicit closed-form solution for the optimal portfolio (Theorem 1), something that is usually difficult to achieve for the smooth ambiguity model. Our result might thus serve as an important building stone for more complex models.

Our analysis yields several novel insights.

The optimal demand for the uncertain asset is highly nonlinear in mean excess return, and it remains bounded, even when expected excess returns are large or when the mean variance tends to zero — cases in which a standard CAPM investor would take an unbounded position. Ambiguity aversion therefore acts as a stabilizing force that prevents extreme exposures (Proposition 1).

The explicit results allow us to perform comparative statics with respect to risk aversion, ambiguity aversion, and the parameters of the gamma distribution. Portfolio demand is decreasing in ambiguity and risk aversion if mean excess return is positive, as is intuitive. On the other hand, when ambiguity aversion grows large, portfolio demand converges to zero. The investor stays away from the uncertain asset as a maxmin investor would do in our model according to Dow & Werlang (1992) (Proposition 2).

We compare our results to a naïve CAPM investor who would simply replace the variance term in the CAPM demand by its Bayesian mean. We show that for small mean excess returns, the demand is well approximated by the naïve CAPM solution, while it is extremely different from the linear CAPM behavior for large excess returns.

The second-order variance of variance yields interesting comparative statics. If we hold the variance of variance fixed while letting the mean of the variance approach zero, a naïve CAPM investor would take an unbounded position in the asset, viewing it as nearly an arbitrage. By contrast, an ambiguity-averse investor maintains a finite, bounded exposure even as the average variance converges to zero and the excess return remains positive (Proposition 4).

By integrating smooth ambiguity preferences with a Variance-Gamma return process, our framework bridges the gap between robust portfolio choice and empirically realistic return modeling. It highlights how uncertainty about volatility—not merely risk itself—

can shape optimal investment behavior in fundamental and economically intuitive ways.

Literature. The Markowitz-CAPM approach of Markowitz (1952), Markowitz et al. (1956) and Merton (1972) is formulated under conditions of risk. The model assumes investors prefer higher returns and lower risk. The expected utility is approximated as a function of the mean and variance under the assumptions outlined in Levy & Markowitz (1979) and Markowitz (2014), forming the basis of the mean–variance approach. Savage (1954) suggests that an investor can assign a unique probability distribution to asset returns. Within this framework, portfolio selection is typically formulated as an expected utility maximization problem, in which the investor selects the optimal allocation of wealth across available assets (see also Tobin (1958), Back (2010), and Quiggin (1993)).

As a limitation, the CAPM investor ignores the possibility of multiple plausible probability distributions. If probabilities cannot be specified for all possible asset returns, the expected utility framework fails to provide a well-defined description of choices under ambiguity. A distinct form of uncertainty, known as ambiguity, arises from not knowing the true distribution (Knight 1921).

In response, economists have developed decision-making models that incorporate ambiguity in portfolio selection. Dow & Werlang (1992), Garlappi et al. (2007) and Vigna (2011) examined portfolio selection under the max–min expected utility framework originally proposed by Gilboa & Schmeidler (1989), in which the investor evaluates each portfolio by the worst-case expected utility over the possible distributions. The smooth ambiguity model, however, allows gradual responses to ambiguity and offers a clear advantage in tractability, as it does not require kinked indifference curves. The model offers the max–min expected utility as a special case, and preserves the expected-utility results, making it straightforward to apply in the presence of ambiguity due to its similar functional form.

The static portfolio selection problem under the smooth ambiguity model has been studied for assets with unknown parameters. Taboga (2005) derives the first order condition for an optimal portfolio. Gollier (2011), Maccheroni et al. (2013), Hara & Honda (2018, 2022) and Mukerji et al. (2023) studied the optimal portfolio for an uncertain asset whose returns are normally distributed with known variance and an uncertain mean, modeled as a normal random variable. Lin & Riedel (2021) and Chen et al. (2025) investigate optimal consumption under Knightian uncertainty and the smooth ambiguity model, respectively, and Asano & Osaki (2020) studies portfolio allocation between a risky asset and an ambiguous asset. He & Hölzermann (2025) uses a two-stage evaluation procedure to study the portfolio selection problem, while Fan et al. (2019) proposed a numerical method to solve this portfolio problem with variance uncertainty. For an overview of the portfolio choice literature under ambiguity, see Guidolin & Rinaldi (2013) and the references therein.

It is well established that the return distributions of most financial assets exhibit semi-heavy tails; in particular, their empirical kurtosis typically exceeds that of the normal distribution (Mandelbrot 1963, Haas & Pigorsch 2010). The Variance–Gamma (VG) distribution, introduced by (Teichroew 1957, Madan & Seneta 1990), is a commonly used fat-tailed model for asset returns, capturing both a pronounced peak near the mean and heavier-than-normal tails while retaining the convenience of a closed-form density. This four-parameter family includes the normal distribution as a limiting case and has finite moments of all orders, as demonstrated in (Gaunt 2023, Fischer et al. 2025). It is thus important to derive the optimal portfolio in this setting.

The remainder of the paper is organized as follows. The next section presents our main result and discusses its economic implications. The final section concludes. All proofs are collected in the appendix.

2 Optimal Investment in Variance–Gamma Models

We consider an investor with wealth $w_0 > 0$ who aims to share this wealth between a single uncertain asset, whose asset return x follows an ambiguous distribution, and a risk-free bond that accrues interest at a positive constant rate r.

The ambiguous return. We assume that the investor faces Knightian uncertainty about the asset return's variance x by assigning a family of normal distributions $\Pi = (N(m, V))_{V>0}$, with known[1] mean m and ambiguous variance parameter $V > 0$. We denote by $\mathbb{E}^V$ the expectation under the belief $N(m, V)$.

The investor holds a second-order probability distribution μ over the ambiguous parameter V, which captures the investor's uncertainty about the variance, which we model as a gamma distribution with a probability density function

$$\mu(V) = \frac{\lambda^\alpha}{\Gamma(\alpha)}(V)^{\alpha-1}\exp(-\lambda V)$$

with the shape parameter $\alpha > 0$ and the rate parameter $\lambda > 0$.

Our model is a one-period version of the Variance Gamma model of Madan & Seneta (1990), a widely used asset pricing framework that accounts for the semi-heavy tails observed in empirical return distributions, going beyond the limitations of normality. By mixing a normal distribution with a Gamma prior on the variance, the model generates peaked distributions around the mean and excess kurtosis while retaining finite moments of all orders. It thus provides a convenient and flexible specification for capturing non-normal daily returns.

For later use, we recall that the ex-ante mean $\mathbb{E}^\mu V = \frac{\alpha}{\lambda}$ and ex-ante variance of our ambiguous parameter V $\mathrm{Var}^\mu V = \frac{\alpha}{\lambda^2}$.

Preferences. The investor has constant absolute risk aversion (CARA) with coefficient $\eta > 0$ and Bernoulli utility function

$$u(x) = -\frac{1}{\eta}\exp(-\eta x).$$

[1]Uncertainty over the mean leads to a typical CAPM-like effect for the portfolio demand, see Gollier (2011). We thus fix the parameter of the mean for a clearer exposition of the results.

In addition, the investor exhibits ambiguity aversion, characterized by the ambiguity index

$$\phi(U) = -\frac{(-U)^{1+\gamma}}{1+\gamma},$$

where $\gamma > 0$ denotes the coefficient of relative ambiguity aversion (Gollier 2011).

Optimization Problem. When the investor invests an amount $\theta \in \mathbb{R}$ in the uncertain asset, her terminal wealth is given by

$$W_\theta = \theta x + (w_0 - \theta)r,$$

where x denotes the uncertain return on the uncertain asset and r the risk-free return. The corresponding indirect utility from choosing portfolio θ is

$$U(\theta) = \int_0^\infty \phi\big(\mathbb{E}^V\left[u(W_\theta)\right]\big)\,\mu(V)\,dV.$$

We now proceed to determine the optimal portfolio. In a first step, we show that the investor attains infinitely negative utility if she takes an excessively large long or short position in the uncertain asset. We then compute explicitly the indirect utility inside the interval of portfolios with finite indirect utility.

Proposition 1. *1. The investor's indirect utility from an investment θ is negative infinity outside the bounded interval $(\underline{\theta}, \overline{\theta})$ where $\underline{\theta} = -\sqrt{\frac{2\lambda}{\eta^2(1+\gamma)}}$ and $\overline{\theta} = \sqrt{\frac{2\lambda}{\eta^2(1+\gamma)}}$.*

2. For an investment $\theta \in (\underline{\theta}, \overline{\theta})$, the indirect utility is given by

$$U(\theta) = -\frac{1}{(1+\gamma)\eta^{1+\gamma}} \frac{\lambda^\alpha \cdot \exp\left[-\eta(1+\gamma)\left(rw_0 + \theta(m-r)\right)\right]}{\left(\lambda - \frac{1}{2}\eta^2(1+\gamma)\theta^2\right)^\alpha}. \tag{1}$$

All proofs are in the appendix.

We now provide some intuition for the somewhat counterintuitive result that the investor avoids large positions in the uncertain asset. Higher investment levels amplify the impact of ambiguity about the variance. Ex ante, the return distribution has heavier tails than the normal distribution, meaning that extreme outcomes become more likely. Since the Gamma distribution for the variance decays exponentially, its Laplace transform fails to exist for sufficiently large values of the parameters such that $\lambda \leq \frac{1}{2}\eta^2(1+\gamma)\theta^2$. Consequently, we obtain infinite negative utility for large investments.

The formula for indirect utility looks pretty complicated, and it might thus come as a surprise that we can now derive the optimal portfolio explicitly.

Theorem 1. *The optimal portfolio θ^* is given by*

$$\theta^* = \frac{\sqrt{\alpha^2 + 2\lambda(1+\gamma)(m-r)^2} - \alpha}{(1+\gamma)\eta(m-r)}$$

for $m \neq r$, and $\theta^* = 0$ for $m = r$.

In terms of ex ante mean and variance of the ambiguous variance V, we have

$$\theta^* = \frac{\sqrt{\mathbb{E}^\mu[V]^4 + 2\,\mathbb{E}^\mu[V]\,\cdot \mathrm{Var}^\mu(V)\,(1+\gamma)\,(m-r)^2} - \mathbb{E}^\mu[V]^2}{\eta\,(m-r)\,(1+\gamma)\;\mathrm{Var}^\mu(V)}. \tag{2}$$

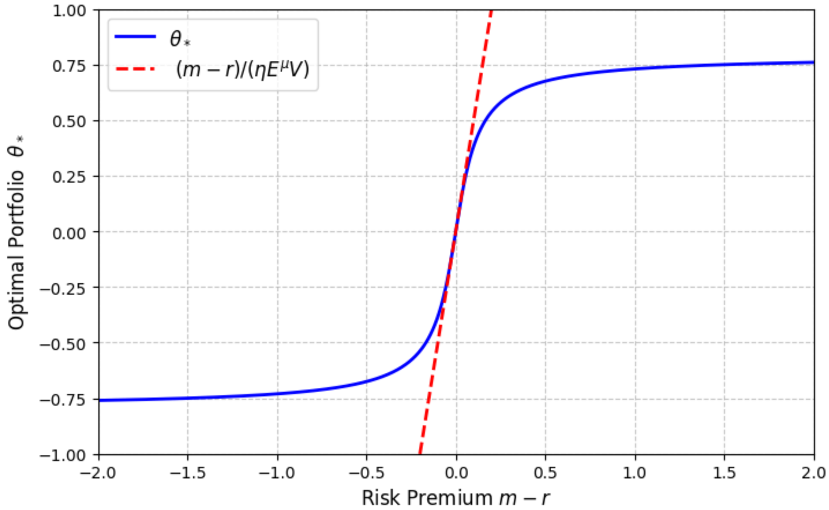

Figure 1: Plot of the optimal demand against mean excess return $m - r$ for parameter values: $\eta = 2,\ \mathbb{E}^\mu[V] = 0.1,\ \mathrm{Var}^\mu(V) = 0.02,$ and $\gamma = 3$.

In the following, we analyze the properties of portfolio demand that result from ambiguity about the return's variance.

The Capital Asset Pricing Model (CAPM) predicts that optimal portfolio demand is linear in both the mean excess return and the precision (the inverse of the return variance). In contrast, our formulation is highly nonlinear and exhibits a striking property: optimal portfolio demand remains bounded. As illustrated in Figure 1, the overall demand function flattens as the mean excess return increases. Investors refrain from taking larger positions in the ambiguous asset because the fear of severe downside outcomes outweighs the potential gains from a higher expected return. The upper bound is given by

$$\overline{\theta} = \sqrt{\frac{2\lambda}{\eta^2(1+\gamma)}}.$$

Portfolio demand is decreasing in risk aversion η and ambiguity aversion γ, as to be expected. More aversion to risk and ambiguity reduces the total maximal exposure the investor is willing to take. The maximal total demand is increasing in the rate parameter λ as a higher rate implies lighter tails for the return distribution that allow for some larger exposure.

Let us next analyze the role of ambiguity aversion γ. On the other hand, if ambiguity aversion grows, we expect demand to converge to the maxmin portfolio demand, which is zero in our case, by a famous result of Dow & Werlang (1992). The following proposition confirms this intuition.

Proposition 2. *1. For positive excess return $m - r > 0$, portfolio demand θ^* is decreasing in ambiguity aversion γ and in risk aversion η.*

2. The ambiguity-neutral expected utility investor invests

$$\theta^* = \frac{\sqrt{\mathbb{E}_\mu[V]^4 + 2\,\mathbb{E}_\mu[V]\, \cdot\, Var(V)\,(m-r)^2} - \mathbb{E}_\mu[V]^2}{\eta\,(m-r)\; Var(V)}.$$

3. If ambiguity aversion γ tends to infinity, optimal investment θ^* converges to 0, the optimal demand for a maxmin investor:

$$\lim_{\gamma\to\infty} \theta^* = 0\,.$$

We next analyze the portfolio demand for small values of mean excess return and compare it to the demand that one would obtain by naïvely replacing the variance term in the classical CAPM demand formula by its prior mean, i.e.

$$\theta^{CAPM} = \frac{m-r}{\eta\,\mathbb{E}^\mu[V]}.$$

Figure 1 suggests that in this case, the linear CAPM demand seems to approximate well the demand under uncertainty. This is indeed correct, as we show in the next proposition.

Proposition 3. *Portfolio demand is close to the CAPM demand for small values of mean excess return $m - r$:*

$$\theta^* \approx \theta^{CAPM} + o\left((m-r)^3\right).$$

The parameter of ambiguity aversion disappears in the second-order Taylor approximation of the portfolio demand. This suggests that ambiguity aversion is a higher order effect for very small values of mean excess return. The situation is very different when we move away from the safe return, of course.

Let us now examine more closely the role of uncertainty about the variance. Figure 2 displays the demand for the uncertain asset as a function of the average prior variance, while holding the prior variance of the return variance fixed (see Equation 2), under the empirically plausible assumption of a positive mean excess return. The figure also includes the naïve CAPM demand for comparison.

Note that the naïve CAPM investor would allocate an unbounded amount to the risky asset as the variance approaches zero. In contrast, the ambiguity-averse investor invests only a small and finite amount, recognizing the (second-order) uncertainty associated with the unknown parameter V. For large values of the prior mean variance, however, the demand of the ambiguity-averse investor converges closely to the CAPM demand.

Proposition 4. *1. For a fixed second-order variance $\mathrm{Var}^\mu(V)$,*

(a) *optimal demand converges to* 0 *as mean ex ante variance* $\mathbb{E}^\mu V$ *converges to zero.*

(b) *optimal demand is asymptotically close to the naïve CAPM demand as* $\mathbb{E}^\mu[V]$ *tends to infinity.*

2. *For a fixed second-order mean* $\mathbb{E}^\mu V$*,*

(a) *optimal demand is close to naïve CAPM demand as* $\mathrm{Var}^\mu(V)$ *tends to zero.*

(b) *optimal demand converges to* 0 *as* $\mathrm{Var}^\mu(V)$ *tends to infinity.*

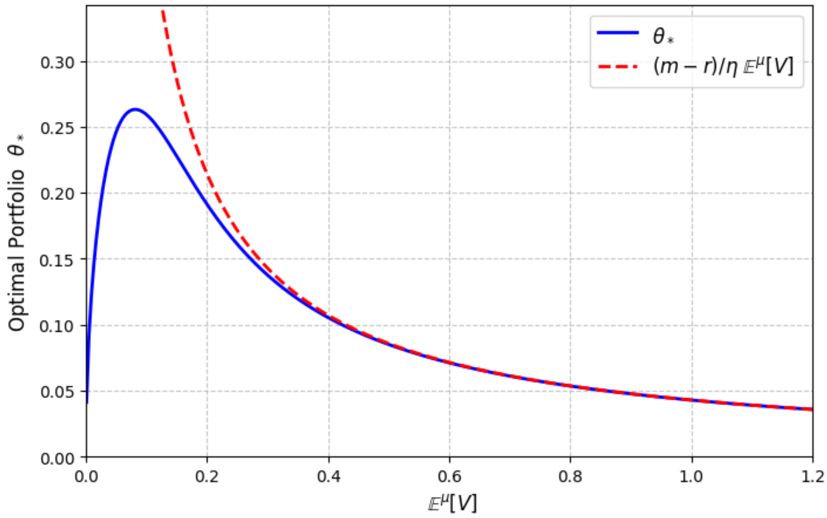

Figure 2: Plot of the optimal demand in terms of average variance $\mathbb{E}^\mu V$ compared to the demand of the naïve CAPM investor, for parameter values $m = 0.08, r = 0.02, \eta = 1.4, \gamma = 2$ and $\mathrm{Var}^\mu(V) = 0.2$.

It may appear counterintuitive that the ambiguity-averse investor refrains entirely from investing in the uncertain asset as the average variance approaches zero, whereas the naïve CAPM investor would take an unbounded position. The explanation lies in the ambiguity-averse investor's recognition of the substantial second-order uncertainty associated with the variance—an aspect that the CAPM investor neglects. She thinks that the asset is almost an arbitrage as the variance is close to zero and we have positive excess return in mean; as a consequence, she would be willing to invest arbitrarily large amounts.

We conclude with discussing the comparative statics with of second-order variance, compare Figure 3. Optimal demand decreases with increasing second-order variance, starting from the naïve CAPM demand. Note that the naïve CAPM investor does not react on the "vol of vol" as the traders call it. She would only care about the mean of the variance, and her exposure thus remains constant in the second-order variance.

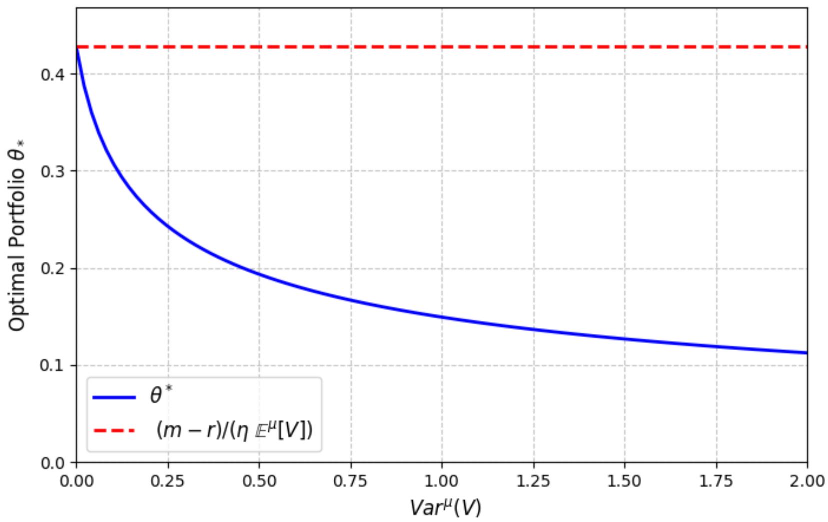

Figure 3: Plot of the optimal demand in terms of variance of ambiguous variance $\mathrm{Var}^{\mu}(V)$ for parameter values: $m = 0.08, r = 0.02, \eta = 1.4, \gamma = 2$ and $\mathbb{E}^{\mu}(V) = 0.1$.

3 Conclusion

We explicitly characterize the optimal investment policy of ambiguity-averse investors in a Variance–Gamma setting. Ambiguity aversion and uncertainty about the variance fundamentally alter behavior relative to a naïve CAPM investor. An investor who recognizes variance uncertainty behaves more cautiously and never chooses excessively large risky positions. The comparative statics differ markedly as well: for small expected returns close to the risk-free rate, ambiguity-averse and naïve investors behave similarly, but their responses to changes in variance diverge sharply. Whereas the CAPM investor is willing to take arbitrarily large positions when variance becomes small and the excess return is positive, the uncertainty-aware investor remains cautious, recognizing the presence of second-order (model) uncertainty. This highlights the importance of accounting for ambiguity in environments with non-Gaussian returns.

A Appendix

A.1 Proof of Proposition 1

Proof. For a portfolio θ with final wealth

$$W_\theta = \theta x + (w_0 - \theta)r,$$

the ex post utility is

$$u(W_\theta) = -\frac{1}{\eta}\exp(-\eta W_\theta) = -\frac{1}{\eta}\exp(-\eta\theta(x-r)) \cdot \exp(-\eta w_0 r).$$

For a given value of the return's variance V, we have $x \sim \mathcal{N}(m, V)$, and the resulting expected utility is

$$\mathbb{E}^V[u(W_\theta)] = -\frac{1}{\eta}\exp(-\eta w_0 r)\cdot \mathbb{E}^V\left[\exp(-\eta\theta(x-r))\right].$$

Using the Laplace transform of the normal distribution we get

$$\mathbb{E}^V\left[\exp(-\eta\theta(x-r))\right] = \exp\left(-\eta\theta(m-r) + \frac{1}{2}\eta^2\theta^2 V\right).$$

Therefore, expected utility given a fixed variance V is

$$\mathbb{E}^V[u(W_\theta)] = -\frac{1}{\eta}\exp\left(-\eta w_0 r - \eta\theta(m-r) + \frac{1}{2}\eta^2\theta^2 V\right).$$

We define $A := \exp\left(-\eta w_0 r - \eta\theta(m-r)\right)$. We next compute the ex ante smooth utility which becomes now a Laplace transform of a Gamma distribution.

$$\begin{aligned}
\mathbb{E}_\mu[\phi(U)] &= -\frac{1}{1+\gamma}\left(\frac{A}{\eta}\right)^{1+\gamma}\int_0^\infty \exp\left(\tfrac{1}{2}\eta^2\theta^2(1+\gamma)V\right)\cdot\frac{\lambda^\alpha}{\Gamma(\alpha)}(V)^{\alpha-1}\exp(-\lambda V)dV \\
&= -\frac{1}{1+\gamma}\left(\frac{A}{\eta}\right)^{1+\gamma}\cdot\frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty (V)^{\alpha-1}\exp\left(-\left[\lambda - \tfrac{1}{2}\eta^2\theta^2(1+\gamma)\right]V\right)dV.
\end{aligned}$$

The above integral is only finite if $\lambda > \frac{1}{2}\eta^2\theta^2(1+\gamma)$, otherwise, the smooth utility is negative infinity, proving the first part of the proposition.

For $\lambda > \frac{1}{2}\eta^2\theta^2(1+\gamma)$, we apply the formula for the Laplace transform of a Gamma distribution to obtain

$$\mathbb{E}_\mu[\phi(U)] = -\frac{1}{1+\gamma}\left(\frac{A}{\eta}\right)^{1+\gamma}\cdot\left(\frac{\lambda}{\lambda - \frac{1}{2}\eta^2\theta^2(1+\gamma)}\right)^\alpha.$$

Plugging $A = \exp\left(-\eta w_0 r - \eta\theta(m-r)\right)$ back in, we get

$$\begin{aligned}
\mathbb{E}_\mu[\phi(U)] &= -\frac{1}{1+\gamma}\left(\frac{\exp\left(-\eta w_0 r - \eta\theta(m-r)\right)}{\eta}\right)^{1+\gamma}\cdot\left(\frac{\lambda}{\lambda - \frac{1}{2}\eta^2\theta^2(1+\gamma)}\right)^\alpha \\
&= -\frac{1}{(1+\gamma)\eta^{1+\gamma}}\frac{\lambda^\alpha\cdot\exp\left[-\eta(1+\gamma)\left(rw_0 + \theta(m-r)\right)\right]}{\left(\lambda - \frac{1}{2}\eta^2(1+\gamma)\theta^2\right)^\alpha}.
\end{aligned}$$

□

A.2 Proof of Theorem 1

Proof. The smooth utility function is concave and wealth W_θ is linear in θ, hence the indirect utility $U(\theta)$ is concave in θ. It thus suffices to check the first-order condition $U'(\theta) = 0$. From Proposition 1,

$$U(\theta) = -\frac{\lambda^\alpha}{(1+\gamma)\eta^{1+\gamma}}\cdot\frac{\exp\left[-\eta(1+\gamma)\left(rw_0 + \theta(m-r)\right)\right]}{\left(\lambda - \frac{1}{2}\eta^2(1+\gamma)\theta^2\right)^\alpha}.$$

Let $C = -\dfrac{\lambda^\alpha}{(1+\gamma)\eta^{1+\gamma}}$, $f(\theta) = \exp[-\eta(1+\gamma)\big(rw_0 + \theta(m-r)\big)]$, and

$$g(\theta) = \frac{1}{\big(\lambda - \frac{1}{2}\eta^2(1+\gamma)\theta^2\big)^\alpha}.$$

Marginal indirect utility is

$$U'(\theta) = C\left(f'(\theta)g(\theta) + f(\theta)g'(\theta)\right),$$

where

$$f'(\theta) = \left[-\eta(1+\gamma)(\mu - r)\right] f(\theta), \qquad g'(\theta) = \frac{\alpha\eta^2(1+\gamma)\theta}{\lambda - \frac{1}{2}\eta^2(1+\gamma)\theta^2} g(\theta).$$

Substituting $f'(\theta)$ and $g'(\theta)$ into the first-order condition, we get

$$0 = U'(\theta) = U(\theta)\cdot\eta(1+\gamma)\left((r-m) + \frac{\alpha\,\eta\,\theta}{\big(\lambda - \frac{1}{2}\eta^2(1+\gamma)\theta^2\big)}\right).$$

As $U(\theta) \neq 0$, we obtain

$$(r-m) + \frac{\alpha\,\eta\,\theta}{\big(\lambda - \frac{1}{2}\eta^2(1+\gamma)\theta^2\big)} = 0.$$

Solving for the optimal portfolio, we get

$$\theta^* = \frac{-\alpha \pm \sqrt{\alpha^2 + 2\lambda(1+\gamma)(m-r)^2}}{\eta(m-r)(1+\gamma)}.$$

The negative solution

$$\theta_* = \frac{-\alpha}{\eta(m-r)(1+\gamma)} - \frac{\sqrt{\alpha^2 + 2\lambda(1+\gamma)(m-r)^2}}{\eta(m-r)(1+\gamma)} \;<\; -\sqrt{\frac{2\lambda}{\eta^2(1+\gamma)}}.$$

lies outside the bound of admissible investments according to Proposition 1. Therefore, we get the unique optimal solution as

$$\theta^* = \frac{\sqrt{\alpha^2 + 2\lambda(1+\gamma)(m-r)^2} - \alpha}{\eta(m-r)(1+\gamma)}.$$

We finally express the optimal portfolio in terms of the mean and variance of the ambiguous variance. Consider the gamma distribution, which models the variance, with mean

$$\mathbb{E}^\mu[V] = \frac{\alpha}{\lambda} \quad \text{and} \quad \mathrm{Var}(V) = \frac{\alpha}{\lambda^2}.$$

Solving for λ and α, we obtain

$$\alpha = \frac{\mathbb{E}^\mu[V]^2}{\mathrm{Var}(V)}, \qquad \lambda = \frac{\mathbb{E}^\mu[V]}{\mathrm{Var}(V)}.$$

Substituting these expressions into the optimal portfolio formula gives

$$\theta^* = \frac{-\alpha + \sqrt{\alpha^2 + 2\lambda(1+\gamma)(m-r)^2}}{\eta(m-r)(1+\gamma)}$$
$$= \frac{-\dfrac{\mathbb{E}^\mu[V]^2}{\mathrm{Var}(V)} + \sqrt{\dfrac{\mathbb{E}^\mu[V]^4}{\mathrm{Var}(V)^2} + \dfrac{2\,\mathbb{E}^\mu[V]\,(1+\gamma)\,(m-r)^2}{\mathrm{Var}(V)}}}{\eta(m-r)(1+\gamma)}.$$

Extracting $1/\mathrm{Var}(V)$ from the square root yields the simplified form:

$$\theta^* = \frac{\sqrt{\mathbb{E}^\mu[V]^4 + 2\,\mathbb{E}^\mu[V]\,\mathrm{Var}(V)\,(1+\gamma)\,(m-r)^2} - \mathbb{E}^\mu[V]^2}{\eta\,(m-r)\,(1+\gamma)\,\mathrm{Var}(V)}.$$

□

A.3 Proof of Proposition 2

1. We first show that portfolio demand is decreasing in ambiguity and risk aversion. Taking derivatives in the optimal portfolio as given in Theorem 1, we obtain

$$\frac{d\theta^*}{d\gamma} = \frac{\alpha\sqrt{\alpha^2 + 2\lambda(m-r)^2(1+\gamma)} - \alpha^2 - \lambda(m-r)^2(1+\gamma)}{\eta(m-r)(1+\gamma)^2\sqrt{\alpha^2 + 2\lambda(m-r)^2(1+\gamma)}} < 0,$$

where we use $\sqrt{\alpha^2 + 2\lambda(m-r)^2(1+\gamma)} > \alpha$. Similarly, we obtain for risk aversion,

$$\frac{d\theta^*}{d\eta} = \frac{\alpha - \sqrt{\alpha^2 + 2\lambda(1+\gamma)(m-r)^2}}{\eta^2(m-r)(1+\gamma)} < 0.$$

2. If the ambiguity aversion γ is 0, then the ambiguity-neutral investor has the demand

$$\theta^* = \frac{\sqrt{\mathbb{E}_\mu[V]^4 + 2\,\mathbb{E}_\mu[V]\,\cdot \mathrm{Var}(V)\,(m-r)^2} - \mathbb{E}_\mu[V]^2}{\eta\,(m-r)\,\mathrm{Var}(V)}.$$

3. The maxmin expected utility paradigm represents ambiguity aversion by assuming the decision maker uses a pessimistic or worst-case evaluation across a set of plausible probability distributions. The first-order expected utility of final wealth,

$$\mathbb{E}^V[u(W_\theta)] = -\frac{1}{\eta}\exp\left(-\eta w_0 r - \eta\theta(m-r) + \frac{1}{2}\eta^2\theta^2 V\right),$$

decreases with an increase in the variance. Therefore, with an ambiguous variance, the max-min investor chooses the normal distribution with the largest possible variance $V_{\max}$ inside the admissible uncertainty set. The worst case for a maxmin investor is an infinite variance, leading to a demand of zero.

For our ambiguity-averse investor, if ambiguity aversion $\gamma \to \infty$, we have

$$\theta^* \approx \frac{\sqrt{2\mathbb{E}_\mu[V]}}{\eta\sqrt{\mathrm{Var}(V)}} \cdot \frac{1}{\sqrt{(1+\gamma)}} \cdot \frac{|m-r|}{(m-r)} \to 0.$$

A.4 Proof of Proposition 3

Proof. We set $z = \frac{2\lambda(1+\gamma)(m-r)^2}{\alpha^2}$. Apply the Taylor expansion for $\sqrt{1+z}$ in order to obtain

$$\begin{aligned}\sqrt{\alpha^2 + 2\lambda(1+\gamma)(m-r)^2} &= \alpha\,(1+z)^{1/2} \\ &= \alpha\left(1 + \tfrac{1}{2}z + O\big(z^2\big)\right) \\ &= \alpha\left(1 + \tfrac{1}{2}\cdot\frac{2\lambda(1+\gamma)(m-r)^2}{\alpha^2} + O\big((m-r)^4\big)\right) \\ &= \alpha + \frac{\lambda(1+\gamma)(m-r)^2}{\alpha} + O\big((m-r)^4\big)\,.\end{aligned}$$

Substituting back into the optimal portfolio, we obtain

$$\theta^* = \frac{-\alpha + \alpha + \frac{\lambda(1+\gamma)(m-r)^2}{\alpha} + O\left((m-r)^4\right)}{\eta(m-r)(1+\gamma)} = \frac{\lambda(m-r)}{\eta\alpha} + o\left((m-r)^3\right).$$

□

A.5 Proof of Proposition 4

Consider the optimal portfolio

$$\theta^* = \frac{\sqrt{\mathbb{E}_\mu[V]^4 + 2\,\mathbb{E}_\mu[V]\ \cdot \mathrm{Var}(V)\,(1+\gamma)\,(m-r)^2}\ - \mathbb{E}_\mu[V]^2}{\eta\,(m-r)\,(1+\gamma)\,\mathrm{Var}(V)}. \tag{3}$$

1. Fix $\mathrm{Var}^\mu(V) > 0$.
   (a) The optimal portfolio expression is continuous in $\mathbb{E}^\mu V$ and converges to zero for $\mathbb{E}^\mu V \to 0$.
   (b) For large values of $\mathbb{E}^\mu[V]$, we use the Taylor expansion

   $$\sqrt{x^4 + ax} - x^2 \approx \frac{a}{2x}$$

   at infinity for $x = \mathbb{E}^\mu[V]$ and $a = 2\mathbb{E}^\mu[V]\ \mathrm{Var}^\mu(V)(1+\gamma)(m-r)^2$ to see that

   $$\sqrt{\mathbb{E}^\mu[V]^4 + 2\mathbb{E}^\mu[V]\ \mathrm{Var}^\mu(V)(1+\gamma)(m-r)^2} - \mathbb{E}^\mu[V]^2 \approx \frac{\mathrm{Var}^\mu(V)\,(1+\gamma)(m-r)^2}{\mathbb{E}^\mu[V]}.$$

   The optimal portfolio then reduces to

   $$\theta^* \approx \frac{\mathrm{Var}^\mu(V)\,(1+\gamma)(m-r)^2}{\mathbb{E}^\mu[V]\,\eta(m-r)(1+\gamma)\ \mathrm{Var}^\mu(V)} = \frac{m-r}{\eta\,\mathbb{E}^\mu[V]} = \theta^{CAPM}.$$

2. Now fix $\mathbb{E}^\mu[V] > 0$.

(a) For small values of $\mathrm{Var}^\mu(V)$, use the first-order expansion $\sqrt{1+\epsilon} \approx 1 + \frac{\epsilon}{2}$ to obtain

$$\begin{aligned}
&\sqrt{\mathbb{E}^\mu[V]^4 + 2\,\mathbb{E}^\mu[V]\,\mathrm{Var}(V)\,(1+\gamma)\,(m-r)^2} - \mathbb{E}^\mu[V]^2 \\
&\qquad = \mathbb{E}^\mu[V]^2 \sqrt{1 + \frac{2(1+\gamma)(m-r)^2\,\mathrm{Var}(V)}{\mathbb{E}^\mu[V]^3}} - \mathbb{E}^\mu[V]^2 \\
&\qquad \approx \frac{(1+\gamma)(m-r)^2\,\mathrm{Var}(V)}{\mathbb{E}^\mu[V]}.
\end{aligned}$$

For the optimal portfolio, divide by the denominator $\eta(1+\gamma)\mathrm{Var}(V)(m-r)$ and you obtain

$$\theta^* \approx \frac{m-r}{\eta\,\mathbb{E}^\mu[V]}.$$

(b) For large $\mathrm{Var}(V)$, the square root function dominates

$$\sqrt{\mathbb{E}^\mu[V]^4 + 2\,\mathbb{E}^\mu[V]\,\mathrm{Var}(V)\,(1+\gamma)\,(m-r)^2} \approx \sqrt{2\,\mathbb{E}^\mu[V]\,(1+\gamma)\,(m-r)^2\,\mathrm{Var}(V)}\,.$$

And hence

$$\theta^* \approx \frac{\sqrt{2\,\mathbb{E}^\mu[V]\,(1+\gamma)\,(m-r)^2\,\mathrm{Var}(V)}}{\eta\,(1+\gamma)\,\mathrm{Var}(V)\,(m-r)} = \frac{\sqrt{2\,\mathbb{E}^\mu[V]}}{\eta\,\sqrt{1+\gamma}} \cdot \frac{1}{\sqrt{\mathrm{Var}(V)}} \cdot \frac{|m-r|}{(m-r)}.$$

We conclude that

$$\theta^* \to 0 \quad \text{as } \mathrm{Var}(V) \to \infty.$$

References

Asano, T. & Osaki, Y. (2020), 'Portfolio allocation problems between risky and ambiguous assets', *Annals of Operations Research* **284**(1), 63–79.

Back, K. (2010), *Asset pricing and portfolio choice theory*, Oxford University Press.

Chen, A., Vanduffel, S. & Wilke, M. (2025), 'Optimal payoffs under smooth ambiguity', *European Journal of Operational Research* **320**(3), 754–764.

Dow, J. & Werlang, S. R. (1992), 'Uncertainty aversion, risk aversion, and the optimal choice of portfolio', *Econometrica: Journal of the Econometric Society* pp. 197–204.

Fan, Y., Teng, S. & Zhang, Q. (2019), 'Analysis of optimal portfolio model under ambiguity', *Journal of Physics: Conference Series* **1168**(5), 052007.

Fischer, A., Gaunt, R. E. & Sarantsev, A. (2025), 'The variance-gamma distribution: A review', *Statistical Science* **40**(2), 235–258.

Garlappi, L., Uppal, R. & Wang, T. (2007), 'Portfolio selection with parameter and model uncertainty: A multi-prior approach', *The Review of Financial Studies* **20**(1), 41–81.

Gaunt, R. E. (2023), 'On the moments of the variance-gamma distribution', *Statistics & Probability Letters* **201**, 109884.

Gilboa, I. & Schmeidler, D. (1989), 'Maxmin expected utility with non-unique prior', *Journal of Mathematical Economics* **18**(2), 141–153.

Gollier, C. (2011), 'Portfolio choices and asset prices: The comparative statics of ambiguity aversion', *The Review of Economic Studies* **78**(4), 1329–1344.

Guidolin, M. & Rinaldi, F. (2013), 'Ambiguity in asset pricing and portfolio choice: A review of the literature', *Theory and Decision* **74**(2), 183–217.

Haas, M. & Pigorsch, C. (2010), *Financial Economics, Fat-Tailed Distributions*, pp. 308–339.

Hara, C. & Honda, T. (2018), 'Implied ambiguity: Mean-variance efficiency and pricing errors', *KIER Discussion Paper* **1004**, 1–45.

Hara, C. & Honda, T. (2022), 'Implied ambiguity: mean-variance inefficiency and pricing errors', *Management Science* **68**(6), 4246–4260.

He, Y. & Hölzermann, J. (2025), 'Portfolio selection and asset pricing with ambiguity: A two-stage evaluation approach', *Available at SSRN 5002924* .

Klibanoff, P., Marinacci, M. & Mukerji, S. (2005), 'A smooth model of decision making under ambiguity', *Econometrica* **73**(6), 1849–1892.

Knight, F. H. (1921), 'Risk, uncertainty and profit', *Hart, Schaffner and Marx* .

Levy, H. & Markowitz, H. (1979), 'Approximating expected utility by a function of mean and variance', *World Scientific-Nobel Laureate Series* p. 281.

Lin, Q. & Riedel, F. (2021), 'Optimal consumption and portfolio choice with ambiguous interest rates and volatility', *Economic Theory* **71**(3), 1189–1202.

Maccheroni, F., Marinacci, M. & Ruffino, D. (2013), 'Alpha as ambiguity: Robust mean-variance portfolio analysis', *Econometrica* **81**(3), 1075–1113.

Madan, D. B. & Seneta, E. (1990), 'The variance gamma (vg) model for share market returns', *Journal of Business* pp. 511–524.

Mandelbrot, B. (1963), 'The variation of certain speculative prices', *The Journal of Business* **36**.

Markowitz, H. (1952), 'Portfolio selection', *The Journal of Finance* **7**(1), 77–91.

Markowitz, H. (2014), 'Mean–variance approximations to expected utility', *European Journal of Operational Research* **234**(2), 346–355.

Markowitz, H. M. et al. (1956), 'The optimization of a quadratic function subject to linear constraints', *Naval research logistics Quarterly* **3**(1-2), 111–133.

Merton, R. C. (1969), 'Lifetime portfolio selection under uncertainty: The continuous-time case', *The Review of Economics and Statistics* **51**(3), 247–257.

Merton, R. C. (1971), 'Optimum consumption and portfolio rules in a continuous-time model', *Journal of Economic Theory* **3**(4), 373–413.

Merton, R. C. (1972), 'An analytic derivation of the efficient portfolio frontier', *Journal of financial and quantitative analysis* **7**(4), 1851–1872.

Mukerji, S., Ozsoylev, H. N. & Tallon, J.-M. (2023), 'Trading ambiguity: a tale of two heterogeneities', *International Economic Review* **64**(3), 1127–1164.

Quiggin, J. (1993), *Generalized expected utility theory*, Springer Science & Business Media.

Savage, L. (1954), *The Foundations of Statistics*, Wiley Publications in Statistics.

Taboga, M. (2005), 'Portfolio selection with two-stage preferences', *Finance Research Letters* **2**(3), 152–164.

Teichroew, D. (1957), 'The mixture of normal distributions with different variances', *The Annals of Mathematical Statistics* **28**(2), 510–512.

Tobin, J. (1958), 'Liquidity preference as behavior towards risk', *The review of economic studies* **25**(2), 65–86.

Vigna, M. D. (2011), Ambiguity made easier, Working Papers - Mathematical Economics 2011-07, Universita' degli Studi di Firenze, Dipartimento di Scienze per l'Economia e l'Impresa.