Leverage Cycles over the Household Life Cycle: A Quantitative Model of Endogenous Leverage and Default

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I construct a quantitative model to rationalize two key features of the U.S. housing market: leverage is positively correlated with housing prices, while mortgage spreads move inversely. In the model, overlapping generations use leverage to accumulate housing, facing a schedule, the Credit Surface, in which interest rates increase with leverage. Negative aggregate shocks depress housing prices as young borrowers default, reducing lenders' wealth and driving up interest rates and spreads. The higher cost of borrowing reduces leverage, feeding back into further declines in housing prices. Crucially, this paper highlights how lenders' risk-bearing capacity endogenously shapes the Credit Surface.

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I. Introduction

Leverage is widely viewed as a key propagation channel through which shocks amplify movements in asset prices and macroeconomic variables. In the context of the housing market, while mortgage loans enable homebuyers to acquire property without a large upfront payment[1], they simultaneously increase households' exposure to housing price volatility and lenders' exposure to default risk. When housing prices decline, highly leveraged households may find themselves underwater and default on their loans. Lenders, suffering from default losses, then demand a higher premium on future loans, which increases the cost of borrowing and further depresses house prices. Understanding these leverage dynamics is therefore crucial for explaining housing booms and busts.

The literature on endogenous leverage, pioneered by Geanakoplos (1997), Geanakoplos and Zame (2014), and Fostel and Geanakoplos (2008, 2014, 2015), establishes a framework where a powerful feedback loop exists between leverage and asset prices. In these models, both the loan-to-value (LTV) limit and equilibrium LTV are determined endogenously and fluctuate in response to shifts in economic fundamentals. However, endogenizing leverage requires modeling a continuum of contracts with distinct leverage levels, each of which must be priced, a task that is both conceptually and computationally challenging. Consequently, much

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[1] This is particularly relevant in the U.S. context, where the vast majority of home purchases are financed through mortgages. According to data from the Federal Reserve Bank of St. Louis (FRED) for the period 1988–2023, mortgages (including conventional, FHA-insured, and VA loans) accounted for an average of 94% of new house sales (Series: HSTFCM, HSTFC, HSTFFHAI, and HSTFVAG).

of this literature remains theoretical, with relatively few quantitative applications. To fill this gap, this paper develops a general equilibrium model with endogenous leverage within an overlapping-generations (OLG) framework. Qualitatively, the model rationalizes two key features of the U.S. housing market: leverage is positively correlated with house prices, while mortgage spreads are negatively correlated with both leverage and housing prices. Quantitatively, the model is calibrated to match the 10–percentage-point decline in the average LTV of first-time homebuyers observed from peak to trough during the Great Recession.

The empirical relationships among leverage, housing prices, and mortgage spreads are documented in Table 1. Using quarterly data from 1999Q1 to 2024Q1, Table 1 reports the cross-correlations of changes in the Case-Shiller housing price index with the average first-time homebuyers' LTV, Combined LTV (CLTV), and mortgage spreads, with the highest absolute correlations underlined. Both LTV and CLTV are positively correlated with housing prices, whereas mortgage spreads are negatively correlated with housing prices. These patterns were particularly pronounced in the 2000s, when the U.S. housing market experienced a leverage cycle. The left panel of Figure 1 illustrates trends in housing prices and CLTV for first-time homebuyers in the U.S. From 1998 onward, CLTV increased steadily along with rising housing prices before experiencing a sharp 10-percentage-point drop from its peak in 2007Q2 to its trough within two years[2]. Mortgage spreads measure the relative cost of purchasing housing assets using leverage. The right panel of Figure 1 shows that mortgage spreads declined modestly from 2% in 1999Q1 to 1.5% in 2007Q2, before surging to 2.61% within a year by 2008Q2[3].

TABLE 1—CROSS CORRELATIONS OF LEVERAGE AND MORTGAGE SPREADS WITH HOUSING PRICES

	LTV	CLTV	Spread
lead (2)	0.41	0.35	-0.08
lead (1)	0.42	0.39	-0.24
contemporaneous	0.41	0.42	-0.28
lag (1)	0.38	0.42	<u>-0.29</u>
lag (2)	<u>0.53</u>	<u>0.44</u>	-0.08

Note: "Lead/lag (i)" indicates that the variable leads/lags housing prices by i quarter(s). LTV is calculated by dividing the balance of the primary mortgage by the appraised value of the property securing the mortgage. CLTV is calculated by summing the balances of all loans secured by a property and then dividing this total by the appraised value of the property. According to Walentin (2014), the duration of a 30-year fixed-rate mortgage in the U.S. is 7 to 8 years. Therefore, I define the mortgage spread as the difference between the average 30-year fixed-rate mortgage and the 5-year treasury yield.
Source: LTV and CLTV: Freddie Mac Single-Family Loan-Level Dataset. Case-Shiller housing price index: FRED. Mortgage spreads: FRED (data labels: MORTGAGE30US, DGS5).

To study these empirical patterns in a life-cycle setting, the model features an economy with overlapping generations of households, each deriving utility from both housing and non-housing consumption. Households face large aggregate endowment shocks that generate substantial fluctuations in equilibrium housing prices and mortgage interest rates. Agents receive low endowments in early life, which gradually increase and peak in middle age before declining in later stages. This incorporation of the household life cycle is grounded in microdata. As illustrated in Figure 2, there is an evident age-related pattern in the housing market: housing wealth increases with age, then slightly declines, while leverage decreases with age.

[2]Fostel and Geanakoplos (2014) also document this comovement of leverage and housing prices, using data on the average down payment for borrowers below the median in the subprime/Alt-A category.

[3]Similarly, Walentin (2014) and Musso, Neri and Stracca (2011) find that mortgage spreads tend to rise during economic downturns, particularly

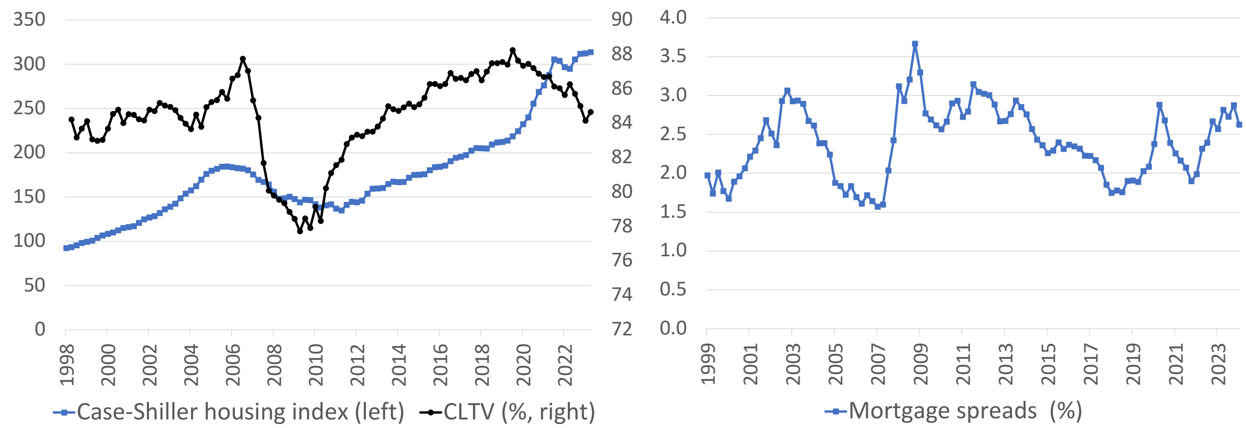

FIGURE 1. HOUSING PRICES, LEVERAGE, AND MORTGAGE SPREADS

Note: Case-Shiller housing price index: January 2000 = 100.

In the model, housing assets are perfectly durable and serve as collateral for borrowing. Agents can accumulate housing wealth using leverage through the simultaneous purchase of housing and issuance of debt contracts. There exists a continuum of debt contracts, each characterized by a different promised repayment and hence a different level of leverage. When the promised repayment exceeds the value of the underlying housing collateral, borrowers default and lenders seize and liquidate the collateral at the prevailing housing price. As a result, risky debt contracts deliver state-contingent payoffs. A collateral constraint prohibits households from issuing a quantity of debt contracts that exceeds their housing stock. The leverage of a contract is measured by its LTV, defined as the ratio of the loan amount to the value of the collateral. LTV ranges from 0% to an endogenously determined upper limit that remains strictly below 100%. Following Fostel and Geanakoplos (2015), equilibrium prices define a "Credit Surface," which represents a menu of leverage levels paired with their corresponding interest rates. The interest rate is weakly increasing in LTV. Loans with sufficiently low LTV that rule out default are priced at the risk-free rate and lie on the flat segment of the credit surface, whereas loans that imply default risk face higher interest rates as LTV increases, corresponding to the upward-sloping segment of the credit surface.

The key feature that distinguishes this paper from the existing quantitative housing literature is the presence of the Credit Surface, which is essentially a risk-based pricing schedule of leverage. Conventional models typically adopt a borrowing constraint following the framework of Kiyotaki and Moore (1997)[4], in which debt is constrained by an exogenously fixed LTV cap. In such frameworks, households borrow by issuing a single riskless bond up to the specified cap, and default is ruled out by construction. As a result, for a given collateral value, all feasible choices of leverage are subject to a single risk-free rate. While the supply side still determines the equilibrium risk-free rate jointly with the demand side, it does not price leverage on

during the Great Recession.

[4]See, for example, Kiyotaki and Moore (1997), Iacoviello (2005), Iacoviello and Neri (2010), Favilukis, Ludvigson and Van Nieuwerburgh (2017), Justiniano, Primiceri and Tambalotti (2019), and Kaplan, Mitman and Violante (2020).

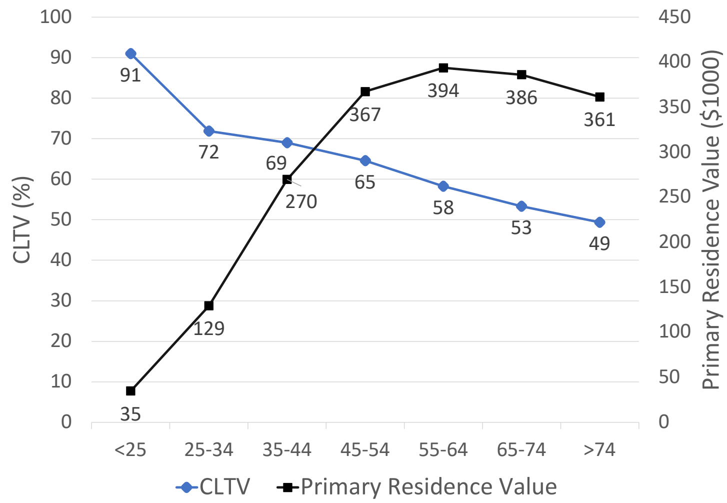

FIGURE 2. LIFE-CYCLE TRENDS

Note: CLTV for the first age group (<25) is averaged based on loans taken for home purchases, and the CLTV for all other age groups is averaged based on loans taken for refinancing purposes. Primary residence values are in 2022 dollars. These patterns are consistent across years, see supplementary materials for details.
Source: CLTV: 2023 HDMA national loan-level dataset. Primary residence values: 2022 Survey of Consumer Finances dataset.

the margin. Such models do not provide a rationale for the determination of leverage and are therefore also silent on why margins change with economic fundamentals. This limits their ability to study how policies or shocks that affect lenders' risk-bearing capacity translate into changes in leverage and, through leverage, into amplified movements in asset prices.

In contrast, the supply side in this paper plays a more active role in the determination of debt by shaping the Credit Surface. Rather than providing a single riskless asset, lenders actively price a continuum of contracts. Because contract repayments are capped by collateral values in each state, the maximum LTV is determined in equilibrium by lenders' state-contingent valuation of the collateral. While agents have a continuous menu of leverage options, only a small subset that maximizes agents' utility is traded in equilibrium. Borrowers face a trade-off between present liquidity and future obligations, where higher leverage provides greater liquidity but comes with higher interest rates to compensate lenders for the expected losses in default states. Ultimately, the market clearing of contracts leaves most LTV levels with zero supply and demand in equilibrium.

In the equilibrium analyzed in this paper, only two types of debt contracts are actively traded: a risky contract, in which agents default during downturns, and a riskless contract, which carries no default risk but offers less liquidity. The model has two main implications. First, households begin with high leverage and gradually reduce it over their lifetime. Young households leverage their housing purchases by borrowing from older cohorts, making them particularly vulnerable to housing price fluctuations. Older households, in contrast, are exposed to housing market risk both directly through housing wealth and indirectly through their holdings of defaultable mortgage contracts. Second, large aggregate endowment shocks generate leverage cycles. In normal states, housing prices and leverage are high, while mortgage rates and spreads remain low. Conversely, during downturns, housing prices and leverage drop sharply, whereas mortgage interest rates and spreads surge.

Life cycle implications. During the early stages of the life cycle, young households receive low endowments. Anticipating future income growth, they borrow through risky contracts and pledge their entire housing stock as collateral. Households then transition to using a combination of risky and riskless contracts for borrowing, no longer needing to pledge all their housing as collateral. As endowments peak and begin to decline in later middle age, they start to invest in risky contracts while simultaneously leveraging their own housing assets with riskless debt. In the final stage, driven by a strong saving motive, households accumulate financial wealth and diversify their portfolios by purchasing both risky and riskless debt contracts. On average, younger households are the most leveraged, and leverage declines with age, a pattern consistent with the data.

Leverage cycle implications. The second main implication of the model is that leverage and housing prices decline substantially during crises, while mortgage rates and spreads surge. A large negative endowment shock reduces housing prices by affecting the two groups of marginal buyers of housing assets, young borrowers and middle-aged lenders, through distinct but related channels. The sequence of events unfolds as follows: during downturns, young borrowers who previously issued risky contracts find themselves underwater and default, which erodes lenders' wealth. Since lenders are marginal buyers of housing assets, their loss of wealth directly depresses housing prices. Lenders, facing significant losses from defaults and declining housing wealth, become reluctant to forgo current consumption to lend. Consequently, they demand higher returns and collectively raise interest rates across all debt contracts, as well as spreads between risky and riskless contracts. The Credit Surface rises and steepens, reducing the leveraging capacity of housing assets for young borrowers. This reduced leverage then feeds back into further declines in housing prices. Figure 3 illustrates the feedback loop between leverage and housing prices in downturns.

Note that the prices of the two contracts fall disproportionately in downturns. The price of the riskless contract remains relatively stable, whereas the price of the risky contract falls sharply. Intuitively, in a downturn lenders expect relatively high consumption in the next-period good state, which is precisely the state in which the risky contract delivers more than the riskless contract. Because lenders anticipate lower marginal utility in that state, they discount these payoffs more heavily. As a result, the risky contract experiences a larger price decline than the riskless contract, reducing its leverage and increasing its implied interest rate. This asymmetric price response steepens the Credit Surface and widens the spread.

In equilibrium, the majority of debt in the economy is issued through the risky contract. Because its payoff varies significantly across states, lenders' wealth fluctuates substantially. As marginal buyers in the housing market, these wealth fluctuations greatly amplify housing price volatility, a channel ignored by models following the Kiyotaki and Moore (1997) tradition.

Solving for the equilibrium is significantly more demanding than in standard quantitative macro frameworks. This paper addresses these challenges and makes a technical contribution by developing an algorithm to compute equilibria with a continuum of contract markets and a high degree of agent heterogeneity. The algorithm can be applied to similar quantitative models that study endogenous leverage. The first challenge is conceptual. There exists an infinite number of potential debt contract markets, and it is ex-ante unclear which of these contracts are selected in equilibrium. It is impractical to derive first-order conditions for all such markets. To address this challenge, I assume that an equilibrium exists in which only a finite subset of contracts is traded, and I then verify that agents have no incentive to deviate by trading contracts with LTV levels outside this finite set.

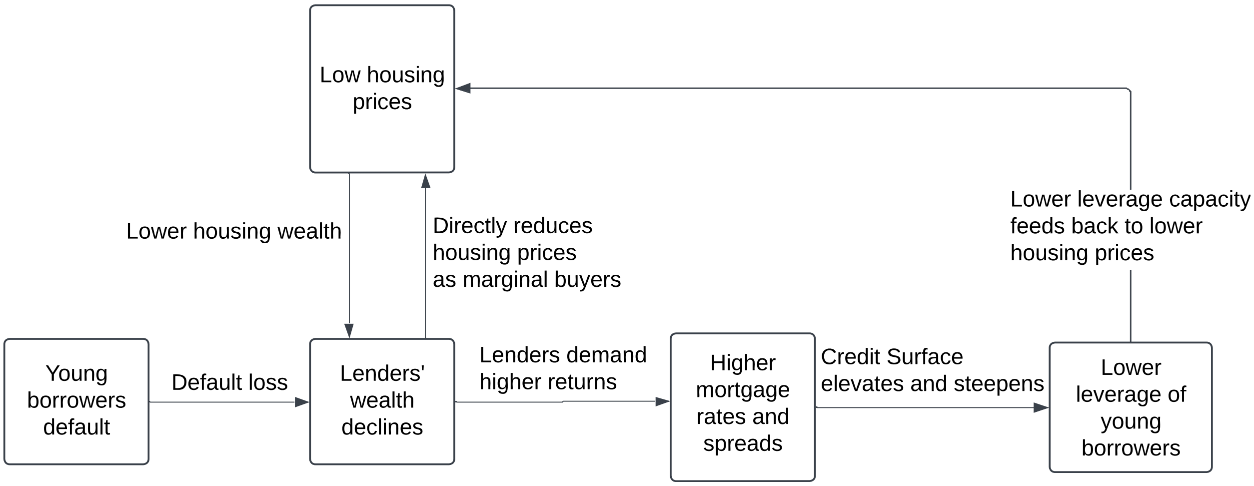

FIGURE 3. THE FEEDBACK LOOP BETWEEN LEVERAGE AND HOUSING PRICES IN DOWNTURNS

The second challenge is computational in nature. The significant heterogeneity among agents and the multiplicity of tradable assets result in a high-dimensional state space. Moreover, the collateral constraint introduces non-linearity into the policy functions. Azinovic, Gaegauf and Scheidegger (2022) introduce a methodology that uses neural networks to address the curse of dimensionality and the non-linearity of policy functions. Following their approach, I use a neural network with three hidden layers to directly approximate the policy functions. The resulting maximum error across all equilibrium conditions is less than 1%.

A. *Related Literature*

This paper bridges two strands of the literature that have been discussed in the introduction: endogenous leverage theory, which emphasizes the feedback loop between leverage and asset prices but has seen relatively few quantitative implementations, and quantitative macroeconomic models of housing, which are empirically disciplined but abstract from risk-based pricing of leverage. This section reviews related quantitative work in each strand to illustrate how existing models either generate different implications within frameworks of endogenous leverage, or abstract from endogenous leverage and rely on alternative mechanisms to generate large housing price fluctuations.

There are two notable quantitative works within the endogenous leverage literature. Diamond and Landvoigt (2022) develop an OLG model where a financial intermediary prices mortgage contracts, but the dynamics are driven primarily by a "deposits-in-utility" channel. In their framework, a standard productivity recession increases households' demand for safe deposits, which expands intermediary funding and lowers mortgage spreads, causing leverage to rise rather than fall. To generate a decline in leverage, they must introduce an idiosyncratic house-value dispersion shock. In contrast, safe-asset demand in my model arises endogenously from life-cycle saving and risk-sharing within a household-to-household credit market. Here, aggregate downturns alone generate deleveraging through the equilibrium pricing of default risk. Consequently, my model delivers the positive comovement between leverage and house prices observed in the data without requiring auxiliary shocks.

Brumm et al. (2015) examine the effects of endogenous leverage on asset price volatility in a model with two infinitely lived agents who differ in risk aversion, where collateral does not provide direct utility. In contrast, this paper emphasizes life-cycle heterogeneity and treats housing as a dual-purpose asset that provides both service flows and collateral. Unlike purely financial assets, housing enters households' utility; consequently, preferences over housing influence equilibrium prices and leverage dynamics.

In the macroeconomic literature on housing that explicitly incorporates household life cycle dynamics, Kaplan, Mitman and Violante (2020) analyze the housing boom and bust episode within an OLG model with endogenous housing prices. They conclude that exogenous shifts in the LTV limit do not significantly affect housing prices; instead, fluctuations in housing prices are primarily driven by shifts in households' beliefs about future housing demand. Favilukis, Ludvigson and Van Nieuwerburgh (2017) also examine the housing market in an OLG model with endogenous housing prices. They find that a higher LTV cap significantly increases housing prices, but this effect is significant only when they introduce heterogeneity in households' bequest preferences, which results in a substantial number of constrained households in equilibrium.

II. Model

A. *Agents, Commodities and Uncertainty*

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. In each period, one of two possible exogenous aggregate shocks $z_t \in \boldsymbol{Z} = \{U, D\}$ realizes. The state U stands for "Up", representing normal times. The state D stands for "Down", representing rare crisis states similar to the Great Recession. The aggregate shock z_t influences both the aggregate endowment and the allocation of endowments among different age groups in every period. z_t evolves according to a Markov chain with transition matrix Γ. Let $\gamma_{z_t,z_{t+1}}$ denote the probability of transitioning from state z_t to state z_{t+1}.

In each period, a continuum of mass 1 identical agents of a new generation is born and lives for A periods. There is no mortality risk; all households die after age A. Age is indexed by $a \in \boldsymbol{A} = \{1, \ldots, A\}$. At the beginning of each period, all households receive a strictly positive endowment of the consumption good, which depends on their age and the aggregate shock: $e_t^a = e^a(z_t) > 0, \forall a \in \boldsymbol{A}$. The aggregate endowment is denoted by $\bar{e}(z_t) = \sum_{a=1}^{A} e^a(z_t)$.

Agents can trade both the consumption good (c_t^a) and perfectly divisible houses (h_t^a)[5] . The consumption good is perishable, whereas houses are perfectly durable and in fixed supply of H. Let the spot price of the consumption good be 1, and the per unit housing price be q_t.

B. *Preferences*

The expected lifetime utility of households born at time t is given by

$$U_t = E_t \sum_{a=1}^{A} \beta^{a-1} u^a(c_t^a, h_t^a), \tag{1}$$

[5] I model housing as a divisible, continuous stock, as is common in quantitative housing macro models. This assumption keeps the household problem smooth and avoids the nonconvexities associated with discrete house-size or tenure choices. See, for example, Davis and Heathcote (2005), Iacoviello (2005), Iacoviello and Neri (2010), and Favilukis, Ludvigson and Van Nieuwerburgh (2017).

where $\beta > 0$ is the discount factor, c_t^a and h_t^a represent the amount of the consumption good and the stock of housing at age a, respectively. $u^a(c,h)$ is an age-dependent period utility function. Let $\hat{\boldsymbol{A}} = \{1,\ldots,A-1\}$ be the set of agents excluding those in the last period of life. All agents have Cobb-Douglas utility over consumption and housing, nested within a constant relative risk aversion utility form when at age $a \in \hat{\boldsymbol{A}}$. For agents in $\hat{\boldsymbol{A}}$, each unit of housing yields one unit of housing service flow. I assume that agents only derive utility from consumption in the final period of life. The period utility function is given by

$$u^a(c,h) = \begin{cases} \dfrac{(c^{1-\alpha}h^{\alpha})^{1-\rho}}{1-\rho}, & a \in \hat{\boldsymbol{A}}, \\ \dfrac{c^{1-\rho}}{1-\rho}, & a = A, \end{cases} \tag{2}$$

where $\alpha > 0$ measures the relative share of housing expenditure, and $\rho > 0$ is the coefficient of risk aversion.

C. Debt Contracts

Households enter the economy with neither debt nor assets. Each period, they meet in anonymous and competitive financial markets to trade collateralized debt contracts. All contracts are one-period: if traded at time t, they are settled at time $t+1$. Let J_t denote the set of such contracts available at time t. A contract in J_t is defined by an ordered pair that specifies its promise and collateral requirement, denoted $(j,1)$. $j \in \mathbb{R}_+$ represents a non-contingent promise to deliver j units of the consumption good in the next period, while the number 1 indicates that the promise j must be backed by one unit of housing as collateral. For simplicity, I will refer to contract $(j,1)$ as "contract j" hereafter. J_t contains infinitely many contracts, as j is continuous and unbounded above.

All contracts are non-recourse, meaning that agents can default on their promises without incurring additional penalties beyond losing the collateral they have previously pledged. An agent who sells one unit of contract j defaults on the promise if j exceeds the realized housing price q_{t+1}. Therefore, the delivery of contract j is $min\{j, q_{t+1}\}$.

The price of contract j is denoted by $\pi_{j,t}$. Let $\theta_{j,t}^a \in \mathbb{R}$ denote the number of contracts j traded by an age-a agent. $\theta_{j,t}^a < 0 (> 0)$ indicates the agent is shorting (going long on) contract j, and by doing so, the agent borrows (lends) $|\pi_{j,t}\theta_{j,t}^a|$. When agents buy one unit of housing and finance this purchase by selling one unit of debt contract j, they are effectively making a down payment of $q_t - \pi_{j,t}$ on the house.

The gross interest rate for contract j is defined as the ratio of its promise to its price:

$$R_{j,t} = \frac{j}{\pi_{j,t}}. \tag{3}$$

The LTV for contract j is the ratio of its price to the value of the collateral:

$$LTV_{j,t} = \frac{\pi_{j,t}}{q_t}. \tag{4}$$

While $LTV_{j,t}$ measures the leverage of individual contracts, Fostel and Geanakoplos (2015) propose metrics to assess the leverage of an agent and the economy-wide leverage. Building on their framework, I define these measures as follows:

The leverage of an agent, LTV_t^a, is defined as the ratio of the agent's total debt issuance to the value of the collateral backing this borrowing:

$$LTV_t^a = \frac{\int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj}{q_t \int_{j\in\mathbb{R}_+} max\{-\theta_{j,t}^a, 0\} dj}. \tag{5}$$

LTV_t^a is not well-defined for agents who do not leverage. To measure borrowing relative to an agent's entire housing stock, define the diluted LTV of an agent, $DLTV_t^a$, as follows:

$$DLTV_t^a = \frac{\int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj}{q_t h_t^a}. \tag{6}$$

The aggregate amount of debt in the economy is the sum of all agents' debt issuance across contracts:

$$D_t = \sum_{a\in\boldsymbol{A}} \int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj. \tag{7}$$

To measure the economy-wide level of leverage, define the LTV of housing, LTV_t^h, as the ratio of aggregate debt to the value of housing used as collateral; and the diluted LTV of housing, $DLTV_t^h$, as the ratio of aggregate debt to the total value of housing:

$$LTV_t^h = \frac{\sum_{a\in\boldsymbol{A}} \int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj}{q_t \sum_{a\in\boldsymbol{A}} \int_{j\in\mathbb{R}_+} max\{-\theta_{j,t}^a, 0\} dj}, \tag{8}$$

$$DLTV_t^h = \frac{\sum_{a\in\boldsymbol{A}} \int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj}{q_t H}. \tag{9}$$

Fostel and Geanakoplos (2008) and Geanakoplos and Zame (2014) introduce the concepts of collateral value and liquidity value as measures of the benefits of borrowing using collateral and different contracts. Following their definitions, in this economy, the liquidity value of contract j for an age-a agent at time t, denoted by $LV_{j,t}^a$, is given by the contract's price net of the present value of its delivery, discounted by the agent's stochastic discount factor (SDF). Let Du_x^a denote the derivative of the period utility function u^a with respect to x,

$$LV_{j,t}^a = \pi_{j,t} - E_t\left[\beta \frac{Du_c^{a+1}(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c^a(c_t^a, h_t^a)} min\{j, q_{t+1}\}\right]. \tag{10}$$

D. Constraints

BUDGET CONSTRAINT. — The budget constraint for age-a agents and at time t is given by

$$c_t^a + q_t h_t^a + \int_{j\in\mathbb{R}_+} \theta_{j,t}^a \pi_{j,t} dj \leqslant e_t^a + q_t h_{t-1}^{a-1} + \int_{j\in\mathbb{R}_+} \theta_{j,t-1}^{a-1} min\{j, q_t\} dj. \tag{11}$$

On the left-hand side of the budget constraint, there are expenditures on consumption, housing, and the total amount of borrowing (or lending) through trading debt contracts in the financial market. On the right-hand side, agents receive their endowments, observe the market value of the housing assets bought in the last period, and clear debt associated with contracts traded in the last period.

The last term on the right-hand side accounts for the total deliveries from contracts traded in the previous period. This term can be written as the sum of two components, as illustrated in (12). For all contracts with $j > q_t$, agents default and deliver the value of the collateral, q_t; for contracts with $j \leqslant q_t$, they fulfill their promise and deliver j.

$$\int_{j\in\mathbb{R}_+} \theta_{j,t-1}^{a-1} min\{j, q_t\} dj = \int_{j>q_t} q_t \theta_{j,t-1}^{a-1} dj + \int_{j\leqslant q_t} j\theta_{j,t-1}^{a-1} dj. \tag{12}$$

COLLATERAL CONSTRAINT. — Borrowing through the sale of contracts requires collateral. Since each contract sold short requires one unit of housing as collateral, agents cannot sell more contracts than their current housing stock. Their choices must satisfy the following collateral constraint:

$$\int_{j\in\mathbb{R}_+} max\{-\theta_{j,t}^a, 0\} dj \leqslant h_t^a. \tag{13}$$

The term $max\{-\theta_{j,t}^a, 0\}$ in the integrand filters out the contracts on which agents make a net purchase, as these do not require collateral.

NO SHORT-SELLING CONSTRAINT. — Agents are prohibited from taking short positions in the housing stock,

$$h_t^a \geqslant 0. \tag{14}$$

E. Collateral Equilibrium

Definition 1. *A collateral equilibrium of the collateral economy is a collection of agents' allocations of consumption and housing, their portfolio holdings of debt contracts, as well as the prices of housing and financial contracts for all t*

$$\Big((c_t^a, h_t^a, (\theta_{j,t}^a)_{j\in J_t})_{a\in\mathbf{A}}; q_t, (\pi_{j,t})_{j\in J_t} \Big)_{t=0}^{\infty} \tag{15}$$

such that

1) *Given $(q_t, (\pi_{j,t})_{j\in J_t})_{t=0}^{\infty}$, the choices $((c_t^a, h_t^a, (\theta_{j,t}^a)_{j\in J_t})_{a\in\mathbf{A}})_{t=0}^{\infty}$ maximize households' lifetime utility (1), subject to the budget constraint (11), the collateral constraint (13), and the no short-selling*

constraint (14).

2) *Markets for the consumption good, housing, and all debt contracts in J_t clear:*

$$\sum_{a=1}^{A} c_t^a = \bar{e}(z_t), \tag{16}$$

$$\sum_{a=1}^{A} h_t^a = H, \tag{17}$$

$$\sum_{a=1}^{A} \theta_{j,t}^a = 0, \forall j \in J_t. \tag{18}$$

F. *The Credit Surface*

In each period, the Credit Surface, determined by equilibrium prices, provides a pricing schedule for leverage. Specifically, it maps each contract to its LTV and interest rate. Figure 4 illustrates an example Credit Surface at a given time t. Let q_{t+1}^U and q_{t+1}^D represent the realizations of housing prices in states $z_{t+1} = U$ and $z_{t+1} = D$, respectively, with $q_{t+1}^U > q_{t+1}^D$. The points A and B represent the LTV and interest rates of two contracts promising q_{t+1}^D and q_{t+1}^U, respectively.

The Credit Surface is characterized by a flat segment at lower LTV levels, where loans are riskless, and an upward-sloping segment where loans imply default risk. The following propositions characterize the shape and properties of the Credit Surface.

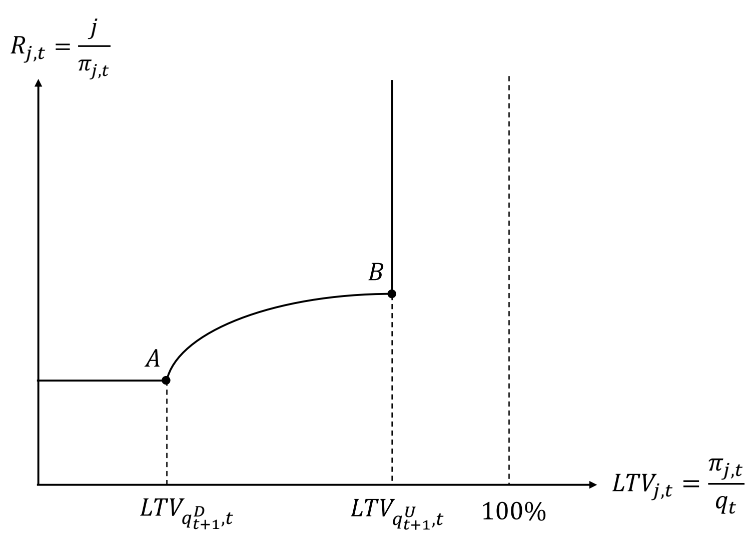

FIGURE 4. A CREDIT SURFACE FOR A BINOMIAL ECONOMY

Proposition 1. *Both $\pi_{j,t}$ and $LTV_{j,t}$ are nonnegative, piecewise strictly increasing functions of j over the interval $[0, q^U_{t+1}]$. For $j \in (q^U_{t+1}, \infty)$, $\pi_{j,t}$ remains constant at $\pi_{q^U_{t+1},t}$ and $LTV_{j,t}$ remains constant at $LTV_{q^U_{t+1},t}$.*

The price of a contract $\pi_{j,t}$ is the expected value of its delivery, discounted by the lender's marginal utility. The delivery $min\{j, q_{t+1}\}$ is non-decreasing in the promise j in every state, and it is strictly increasing in those states in which the housing price exceeds the promise. Since lenders' discount factors are strictly positive, increasing j weakly increases the discounted delivery. Hence, $\pi_{j,t}$ is nonnegative and non-decreasing in j, and strictly increasing whenever the housing price exceeds j. The same monotonicity carries over to $LTV_{j,t}$, since it is proportional to $\pi_{j,t}$. Once j exceeds the maximal housing price q^U_{t+1}, the delivery no longer depends on j. Consequently, both $\pi_{j,t}$ and $LTV_{j,t}$ are fixed at $\pi_{q^U_{t+1},t}$ and $LTV_{q^U_{t+1},t}$. See Appendix A for the formal proof.

Proposition 1 establishes that there is a *strict* mapping from j to $LTV_{j,t}$ over $[0, q^U_{t+1}]$. Note that there is also a strict mapping from j to $R_{j,t}$ over (q^U_{t+1}, ∞). Therefore, each contract j corresponds *uniquely* to a unique point on the Credit Surface. This allows each contract market to be interpreted as a market for leverage at that specific $LTV_{j,t}$. Agents do not merely choose a loan amount, they choose an optimal degree of leverage from an endogenous schedule pinned down by the equilibrium pricing in the contract markets.

Proposition 2. *The relationship between $R_{j,t}$ and $LTV_{j,t}$ is characterized as follows:*

- *For $j \in [0, q^D_{t+1}]$, $R_{j,t}$ is constant over the interval $[0, LTV_{q^D_{t+1},t}]$.*
- *For $j \in (q^D_{t+1}, q^U_{t+1}]$, $R_{j,t}$ is increasing and concave in $LTV_{j,t}$ over $(LTV_{q^D_{t+1},t}, LTV_{q^U_{t+1},t}]$.*
- *For $j \in (q^U_{t+1}, \infty)$, as j goes to infinity, $LTV_{j,t}$ remains constant at $LTV_{q^U_{t+1},t}$ while $R_{j,t}$ goes to infinity.*

The Credit Surface is flat until point A, then rises until point B, after which it becomes a vertical line.

Appendix B provides the formal proof. Proposition 2 establishes that, at a given point in time, the interest rate increases with LTV. For contracts with a promise smaller than q^D_{t+1}, borrowers will not default; hence, such contracts are charged a uniform riskless interest rate. For contracts with a promise between q^D_{t+1} and q^U_{t+1}, borrowers default in state $z_{t+1} = D$ but not in $z_{t+1} = U$. The higher the j, the greater the amount of debt borrowers will default on in the D state; therefore, the interest rate is higher for contracts with a larger j. Lastly, when contracts promise an amount exceeding the housing prices in both states, lenders anticipate that borrowers will default in both states and will not pay more than the price of the contract with $j = q^U_{t+1}$. The interest rate goes to infinity as j goes to infinity and $\pi_{j,t}$ remains constant. This relationship between interest rates and LTV aligns with the empirical findings of Geanakoplos and Rappoport (2019), who estimate Credit Surfaces for 2007Q2 and 2008Q4 and show that interest rates increase with both LTV and credit score.

$R_{j,t}$ is concave in $LTV_{j,t}$ when $j \in (q^D_{t+1}, q^U_{t+1}]$. The concavity implies that as soon as j exceeds q^D_{t+1}, the contract transitions from riskless to one carrying default risk, leading to a steep increase in the interest rate. Borrowers default in $z_{t+1} = D$, providing a small repayment precisely when lenders experience the highest marginal utility. Consequently, lenders become highly sensitive to small increases in leverage and demand a steep increase in the risk premium to compensate for the perceived default risk.

Proposition 3. *The upper limit of $LTV_{j,t}$ is endogenous and is strictly less than 100%.*

Proposition 3 asserts that agents must always make a strictly positive down payment to acquire housing. Unlike pure financial assets, housing provides immediate utility from occupancy in addition to its role as collateral and a store of value. Because the maximum delivery of any contract is capped by the future realization of housing prices, the maximum loan amount lenders provide is limited to the discounted value of that collateral. However, the current housing price must reflect both the households' marginal utility of current occupancy and its expected future value. Consequently, even with the most leveraged contract, a borrower must still pay for the current period's housing service flow; hence, the LTV remains strictly below 100%. If households did not derive utility from housing, the down payment could be zero with the most leveraged contract, and LTV could reach 100%. See Appendix C for the formal proof.

III. Quantitative Analysis

A. *Parameterization*

TABLE 2—PARAMETERS

Parameter	Description	Value
Demographics		
A	Life span	30
Aggregate uncertainty		
Γ	Transition matrix	$\begin{bmatrix} 0.86 & 0.14 \\ 0.77 & 0.23 \end{bmatrix}$
Preferences		
β	Discount factor	0.92
ρ	Risk aversion	5.5
α	Weight on housing	0.115
Age-endowment profiles		
$\{e^a(U)\}_{a=1}^{20}$	In state U	See Figure 5
$\{e^a(D)\}_{a=1}^{20}$	In state D	See Figure 5
Housing		
H	Housing supply	30

DEMOGRAPHICS. — Households live for thirty periods, $A = 30$. In the model, one period corresponds to two calendar years. Households start their economic life at real age 21 (model age $a = 1$), and live until real age 80 (model age $A = 30$). The two-year period length is set to maintain computational feasibility while preserving a realistic life-cycle horizon.

AGGREGATE UNCERTAINTY. — Transition probabilities in Γ are chosen to match two empirical features: (i) the unconditional fraction of time the economy spends in a Great-Recession-like (D) state, and (ii) the average duration of a Down state. Using annual U.S. real GDP per capita from the Maddison Project Database

(2023), spanning 1800–2022, I apply the Hodrick–Prescott filter to log GDP per capita with smoothing parameter $\lambda = 100$ to decompose the series into trend and cyclical components. In 2009, real GDP per capita was 3.252% below its HP trend. I therefore classify a year as being in the crisis state D whenever detrended log GDP per capita falls by more than 3.252% below trend. Under this definition, the U.S. economy is in state D 15.2% of the time, and the average D spell lasts 1.7 calendar years. These two moments pin down the annual transition probabilities and hence the annual Markov transition matrix over states (U,D). Because one model period corresponds to two calendar years, I set the model-period transition matrix to the square of the annual transition matrix.

PREFERENCES. — I choose the discount factor β and the weight on housing α within ranges commonly used in the macroeconomic housing literature. I set $\beta = 0.92$, which corresponds to an annual discount factor of 0.96. I then choose α so that the model delivers a housing expenditure share of 11.5%[6].

The coefficient of relative risk aversion, ρ, governs the curvature of marginal utility and therefore the pricing of state-contingent payoffs. A higher ρ increases the sensitivity of the stochastic discount factor to consumption fluctuations and raises the compensation lenders require for bearing state risk. Holding other parameters fixed, greater risk aversion induces a sharper contraction in credit supply during downturns, thereby widening the dispersion in equilibrium leverage across aggregate states.

While the discount factor and transition probabilities also influence equilibrium pricing, ρ is the power parameter in preferences and therefore has a first-order impact on lenders' risk-bearing capacity. I calibrate ρ to match the 10–percentage-point difference in the average leverage of first-time homebuyers between states U and D, and focus the sensitivity analysis on variations in ρ while holding other parameters fixed. In Appendix D, I present equilibrium outcomes under alternative values of ρ and discuss the role of risk aversion in shaping the magnitude of the leverage cycle.

AGE-ENDOWMENT PROFILES. — Figure 5 displays the life cycle age profiles of endowments for both aggregate states. The Survey of Consumer Finances (SCF) provides detailed data on household income and wealth in the United States. To study the aftermath of the Great Recession, the SCF conducted a special panel survey between 2007 and 2009. Households that participated in the 2007 survey were invited to a follow-up survey in 2009. The income data collected from these households in both years serve as the basis for constructing the endowment age profiles across states in the model.

Specifically, I restrict the sample to households that were homeowners in 2007 and partition them into 30 two-year age bins covering ages 21–80. I then calculate OECD-modified equivalized income, using 2007 household composition data to adjust for household size. The SCF-weighted mean equivalized income within each bin for 2007 and 2009 forms the raw age–endowment profiles for states U and D, respectively. Finally, because the raw profiles are spiky (shown as dotted lines), I smooth them by fitting a cubic polynomial across age bins and use these fitted values as the model's endowment profiles.

The resulting profiles reveal that young agents face a steep upward trajectory in their endowments and maintain high growth regardless of the aggregate state. In contrast, middle-aged agents experience a large decline in their income during the Great Recession. While older agents also see a decline in the recession, the magnitude is much smaller compared to the middle-aged.

[6] Iacoviello (2005) implies a housing share of 9%; Iacoviello and Neri (2010) implies 11%; and Davis and Heathcote (2005) implies 13%.

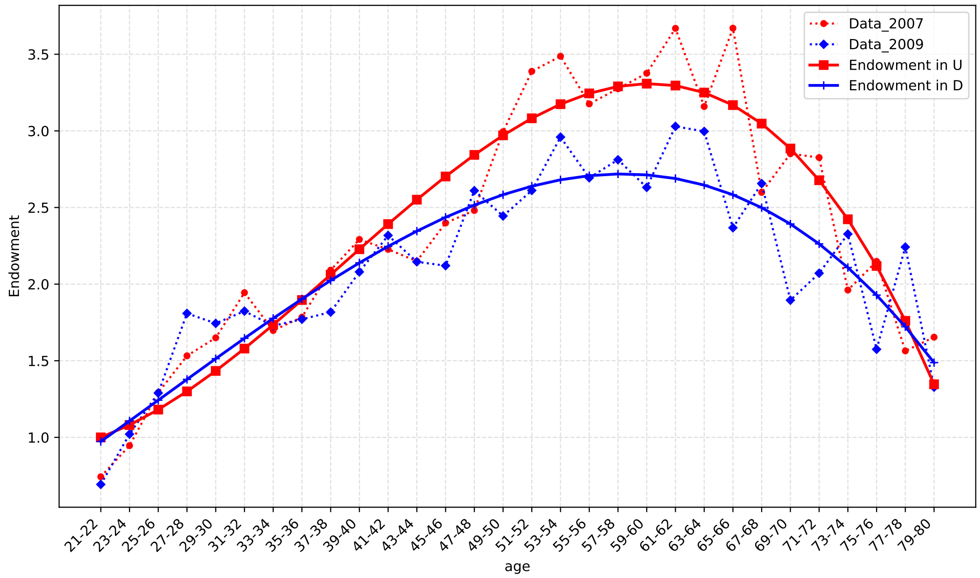

FIGURE 5. LIFE CYCLE AGE PROFILES OF ENDOWMENTS

HOUSING SUPPLY. — I set the aggregate housing supply to $H = 30$ so that, in equilibrium, the average housing holding across agents is around one, which improves numerical stability.

B. Numerical Solutions

In this section, I discuss the challenges of solving the collateral equilibrium. I then define a Functional Rational Expectations Equilibrium (FREE) of the economy, which can be solved by a manageable system of equations. I then prove that a FREE is a genuine collateral equilibrium. Readers primarily interested in the quantitative findings may move directly to Section IV.

CHALLENGES OF LOCATING EQUILIBRIA. — Finding a collateral equilibrium for this economy presents two main challenges: one conceptual and the other computational. Conceptually, it is challenging to derive the optimality conditions for households. Because households make choices over a continuum of contracts, $\int_{j\in\mathbb{R}_+} \theta^a_{j,t} dj$, it is impractical to derive an infinite number of Euler equations with respect to each individual contract $\theta^a_{j,t}$.

According to Geanakoplos (1997), only a limited set of contracts will be traded in equilibrium due to the scarcity of collateral. However, it remains unclear which specific contract(s) will have both non-zero supply and demand in equilibrium. The No-Default Theorem, formalized by Fostel and Geanakoplos (2015), states that in a binomial economy in which financial assets serve as collateral, the contract that maximizes the promised payment without defaulting will be actively traded in equilibrium. However, this theorem does not extend to models in which non-financial assets, such as housing, serve as collateral. A key prerequisite of the theorem is that collateral provides no intrinsic utility to agents, allowing them to freely reshuffle their portfolio holdings to achieve the desired wealth transfer. In this paper, agents cannot freely reshuffle housing holdings because the demand for housing depends not only on its role in transferring wealth across time and states, but also on the intrinsic utility it provides.

Geanakoplos (1997) presents an example with two types of agents and housing as collateral, concluding that, in equilibrium, only the contract promising the highest possible realization of housing prices in the next period would be traded. However, in a more complex setting with 30 types (generations) of agents, such an equilibrium does not exist. Assuming an equilibrium in which only the contract that promises q^U_{t+1} is traded in each period, I solve for variables that satisfy all the Karush-Kuhn-Tucker (KKT) conditions and market clearing constraints under this assumption. The results reveal that agents making net purchases of the contract do not agree on the prices of all other contracts in J_t. This disagreement implies the presence of arbitrage opportunities, indicating that prices have not reached equilibrium and thus contradicting the initial assumption. To address this challenge, I start by guessing an equilibrium regime in which only a finite number of contracts are traded. I then verify whether agents would deviate from it.

Numerically approximating an equilibrium is challenging for two reasons. Firstly, the presence of 30 types of agents trading multiple assets inevitably leads to a large number of state variables. It is also difficult to pin down households' portfolio choices when assets have similar payoff structures, as this creates a near-indeterminacy in the agents' optimization problem. Secondly, the model features occasionally binding collateral constraints, which introduces non-linearity in the policy functions, making them difficult to approximate accurately with linear methods. Azinovic, Gaegauf and Scheidegger (2022) establish that neural networks, combined with approximating the equilibrium only within the ergodic state space, have the potential to address the challenges of high dimensionality and non-linearity while being computationally efficient. Following their approach, I use a neural network with two hidden layers to approximate the equilibrium.

FUNCTIONAL RATIONAL EXPECTATIONS EQUILIBRIUM. — Assuming that an equilibrium exists in which only a finite number of contracts are actively traded with non-zero supply and demand in each period. Let $\hat{J}_t$ denote the set of traded contracts and N_j its cardinality. Under this assumption, contracts not included in $\hat{J}_t$ have zero supply and demand, i.e., $(\theta^a_{j,t})_{\forall a \in \boldsymbol{A}, j \in J_t \setminus \hat{J}_t} = 0$ for all t. This simplifies the budget constraint and reduces the equilibrium conditions to a finite system of KKT and market clearing conditions.

To numerically approximate the candidate equilibrium, I follow Spear (1988) and Krueger and Kubler (2004) to define the Functional Rational Expectations Equilibrium, as described below. I then use a neural network to approximate the policy and pricing functions, ensuring that the equilibrium conditions are satisfied within an acceptable tolerance. Upon solving for these functions, I examine whether agents have incentives to deviate and trade contracts not included in $\hat{J}_t$. Specifically, I verify whether the third and fourth conditions in the definition of FREE are satisfied. If these conditions are met, the process concludes. If not, I adjust the set of actively traded contracts and repeat the steps until all conditions in the definition of a FREE are fulfilled. The algorithm of the approximation process is detailed in Appendix E.

Definition 2. *A FREE of the collateral economy, in which only contracts in $\hat{J}_t$ are traded with non-zero demand and supply, consists of $(h^a_t, \theta^a_t, \mu^a_t)_{a \in \boldsymbol{A}}$ and (q_t, π_t), where:*

- $\theta^a_t = (\theta^a_{j,t})_{j \in \hat{J}_t}$ *denotes agents' portfolio holdings of contracts in $\hat{J}_t$,*
- $\mu^a_t = (\mu^{a,S}_t)_{S \subseteq \hat{J}_t, S \neq \emptyset}$ *denotes the Lagrange multipliers for the collateral constraints corresponding to all the non-empty subsets S of $\hat{J}_t$,*

- $\pi_t = (\pi_{j,t})_{j\in\hat{J}_t}$ *denotes the prices of contracts in* $\hat{J}_t$,

as time-invariant policy functions and pricing functions of x_t*, where*

$$x_t = (z_t, (h^a_{t-1}, \theta^a_{t-1})_{a\in\mathbf{A}}, (\gamma_{z_t,U}, \gamma_{z_t,D})) \in \mathbf{Z} \times [0,H]^A \times ([-H,H]^{N_j})^A \times [0,1]^2$$

is the state vector[7]*, such that the following conditions are satisfied.*

1) *The following complementary slackness conditions hold for* $a \in \hat{\mathbf{A}}$*,* $j \in \hat{J}_t$*, and for any non-empty subset* $S \subseteq \hat{J}_t$*:*

$$\mu^{a,S}_t (\sum_{j\in S} -\theta^a_{j,t} - h^a_t) = 0, \tag{19}$$

$$\sum_{j\in S} -\theta^a_{j,t} - h^a_t \leqslant 0, \tag{20}$$

$$\mu^{a,S}_t \geqslant 0. \tag{21}$$

2) *Define* $\mu_{j,t}$ *as the sum of Lagrange multipliers for all subsets S that include contract j:* $\mu^a_{j,t} \equiv \sum_{S\subseteq\hat{J}_t : j\in S} \mu^{a,S}_t$*. Define* $\mu^a_{h,t}$ *as the sum of Lagrange multipliers for all non-empty subset S:* $\mu_{h,t} \equiv \sum_{S\subseteq\hat{J}_t} \mu^{a,S}_t$*. The following first-order conditions hold for all* $a \in \hat{\mathbf{A}}$*,* $j \in \hat{J}_t$*:*

$$q_t = \frac{Du^a_h(c^a_t, h^a_t)}{Du^a_c(c^a_t, h^a_t)} + E_t\left[\beta \frac{Du^{a+1}_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du^a_c(c^a_t, h^a_t)} q_{t+1}\right] + \frac{\mu^a_{h,t}}{Du^a_c(c^a_t, h^a_t)}, \tag{22}$$

$$\pi_{j,t} = E_t\left[\beta \frac{Du^{a+1}_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du^a_c(c^a_t, h^a_t)} min\{j, q_{t+1}\}\right] + \frac{\mu^a_{j,t}}{Du^a_c(c^a_t, h^a_t)}. \tag{23}$$

Following Fostel and Geanakoplos (2008), I define the fundamental value of housing as the sum of the first two components of q_t *in (22), and the collateral value as the third component.*

3) *The state-dependent intertemporal marginal utilities between* t *and* $t+1$ *are equal among unconstrained agents. Unconstrained agents are defined as those for whom the Lagrange multipliers associated with the collateral constraint are zero, i.e.,* $\mu^a_t = 0$.[8] *Let* $\bar{\mathbf{A}}$ *be the set of unconstrained agents, then for all* $a, b \in \bar{\mathbf{A}}$*,* $z_{t+1} \in \mathbf{Z}$*:*

$$\beta\gamma_{z_t,z_{t+1}} \frac{Du^{a+1}_c(c^{a+1,z_{t+1}}_{t+1}, h^{a+1,z_{t+1}}_{t+1})}{Du^a_c(c^a_t, h^a_t)} = \beta\gamma_{z_t,z_{t+1}} \frac{Du^{b+1}_c(c^{b+1,z_{t+1}}_{t+1}, h^{b+1,z_{t+1}}_{t+1})}{Du^b_c(c^b_t, h^b_t)}. \tag{24}$$

[7] The collateral constraint and the market clearing conditions for contracts restrict θ^a_{t-1} to lie within $[-H,H]^{N_j}$.

[8] In equilibrium, unconstrained agents (with $\mu^a_t = 0$) must agree on the prices of all contracts in J_t. Suppose that two unconstrained agents, a and b, have different state-dependent intertemporal marginal utilities, implying different present values for the same contract. Without loss of generality, assume agent a values a contract higher than agent b, agent a would have an incentive to buy more of it, while b would hold less. This incentive to trade indicates that the market is not yet in equilibrium, contradicting the initial assumption. Therefore, unconstrained agents must agree on $(\pi_{j,t})_{j\in J_t}$.

$(\pi_{j,t})_{j\in J_t \setminus \hat{J}_t}$ *are given by the present value of contracts for unconstrained agents:*

$$\pi_{j,t} = E_t\left[\beta \frac{Du_c^{a+1}(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c^a(c_t^a, h_t^a)} min\{j, q_{t+1}\}\right], a \in \bar{\boldsymbol{A}}. \tag{25}$$

4) *Define constrained agents as those for whom the collateral constraint is binding, with at least one element of μ_t^a strictly positive, i.e., $\mu_t^a \neq 0$ and $\mu_t^a \geqslant 0$. Such agents exclusively issue one optimal contract, which has the highest liquidity value among all contracts in J_t, and its liquidity value equals the collateral value*[9]*. Formally, let $j_t^{a,*}$ denote the optimal contract for age-a agents, and $\tilde{\boldsymbol{A}}$ the set of constrained agents, then for all $a \in \tilde{\boldsymbol{A}}$:*

$$-\theta_{j_t^{a,*},t}^a - h_t^a = 0, \ (\theta_{j,t}^a)_{\forall j \in J_t \setminus \{j_t^{a,*}\}} = 0, \tag{26}$$

$$LV_{j_t^{a,*},t}^a = \frac{\mu_{h,t}^a}{Du_c^a(c_t^a, h_t^a)} \geqslant LV_{j,t}^a, \ \forall j \in J_t \setminus \{j_t^{a,*}\}. \tag{27}$$

5) *The budget constraint is satisfied for all $a \in \boldsymbol{A}$:*

$$c_t^a + q_t h_t^a + \sum_{j\in\hat{J}_t} \theta_{j,t}^a \pi_{j,t} = e_t^a + q_t h_{t-1}^{a-1} + \sum_{j\in\hat{J}_t} \theta_{j,t-1}^{a-1} min\{j, q_t\}. \tag{28}$$

6) *All markets clear:*

$$\sum_{a=1}^{A} c_t^a = \bar{e}(z_t), \tag{29}$$

$$\sum_{a=1}^{A} h_t^a = H, \tag{30}$$

$$\sum_{a=1}^{A} \theta_t^a = 0. \tag{31}$$

Proposition 4. *A FREE for the collateral economy induces a collateral equilibrium.*

PROOF:

To prove that a FREE sufficiently induces a collateral equilibrium, we must establish that, in a FREE, all agents are maximizing their utility subject to the budget constraint, collateral constraint, and the no short-selling constraint, and that all markets clear. The proof proceeds by addressing each condition of a FREE, verifying market clearing and agent optimization with the absence of incentives to deviate.

First, I examine, in a FREE, agents optimize subject to the collateral constraint specified in (13). In a FREE, under the condition that only a finite number of contracts in the set $\hat{J}_t$ are traded with non-zero

[9]As shown by Geanakoplos and Zame (2014), the liquidity value of the optimal contract equals the collateral value. See Appendix F for a proof of this result in the context of a FREE.

supply and demand, the pertinent collateral constraint simplifies to

$$\sum_{j\in\hat{J}_t} max\{-\theta^a_{j,t},0\} \leqslant h^a_t.$$

This condition is equivalent to imposing that for any non-empty subset of $\hat{J}_t$, the negative of the sum of $\theta^a_{j,t}$ across all the contracts in that subset does not exceed the housing stock h^a_t. Mathematically, this yields $\sum_{n=1}^{N_j}\binom{N_j}{n}$ distinct inequalities:

$$\sum_{j\in S} -\theta^a_{j,t} \leqslant h^a_t \quad \text{for all non-empty subsets } S\subseteq \hat{J}_t. \tag{32}$$

By reducing the original constraint (13), which involves an integral and a maximization operator, to a set of standard inequality constraints, we make it possible to formulate the Lagrangian problem for the households. The first condition of a FREE specifies the complementary slackness conditions associated with the inequalities in (32).

The second condition of a FREE corresponds to the first-order conditions of the Lagrangian with respect to households' choices of consumption, housing, and contracts in $\hat{J}_t$. The fifth and sixth conditions ensure that the budget constraint and market clearing conditions are satisfied. The no short-selling constraint is satisfied as households derive utility from housing, and the convexity of the utility function ensures nonnegative consumption and housing holdings.

The final step is to show that, in a FREE, households have no incentive to deviate by trading contracts outside the finite set $\hat{J}_t$. We discuss two groups of households based on whether the collateral constraint is binding.

Unconstrained households. For unconstrained agents, the marginal benefit of borrowing using any contract is zero. Consequently, these agents are indifferent to trading any contracts in J_t as long as the collateral constraint is not violated, and thus, they have no incentive to deviate. The third condition of a FREE asserts the absence of arbitrage opportunities: no agent can make a profit in any state given the prevailing prices.

Constrained households. The fourth condition of a FREE asserts that $\hat{J}_t$ contains an optimal contract $j^{a,*}_t$ for constrained agents, which has the highest liquidity value among all contracts in J_t. Consequently, constrained agents have no incentive to issue contracts outside $\hat{J}_t$.

Liquidity value guides borrowers in selecting among various contracts written against the same collateral, as it quantifies the trade-off between the immediate liquidity provided by a contract and its future obligations. By rearranging the Euler equation (23) and using the definition of $LV^a_{j,t}$, we can see that the liquidity value of contract j is equal to $\mu^a_{j,t}$ divided by agent's current marginal utility of consumption[10]:

$$LV^a_{j,t} = \frac{\mu^a_{j,t}}{Du^a_c(c^a_t,h^a_t)}.$$

As defined earlier, $\mu^a_{j,t}$ is the sum of Lagrange multipliers associated with the collateral constraints for every subset of $\hat{J}_t$ that includes contract j. It represents the additional utility agents would gain if they

[10]For contracts not included in $\hat{J}_t$, their liquidity value can be computed using the definition provided in (10).

could marginally increase the sale of contract j by relaxing all relevant collateral constraints. Dividing $\mu^a_{j,t}$ by $Du^a_c(c^a_t, h^a_t)$ translates this abstract utility gain into a tangible metric: the liquidity value $LV^a_{j,t}$, which quantifies the marginal benefit in terms of real consumption goods.

If $\mu^a_{j,t} > \mu^a_{k,t}$ for all $k \in J_t$ with $k \neq j$, contract j offers the highest marginal benefit per unit of collateral. In this case, agents will allocate collateral exclusively to contract j until the collateral constraint binds, i.e., $-\theta^a_{j,t} = h^a_t$. If $\mu^a_{j,t} = \mu^a_{k,t} > \mu^a_{l,t}$ for $k \in S_k \subset J_t$ and $l \in J_t \backslash \{j, S_k\}$, agents are indifferent between issuing contract j and any contract in S_k and will set $\theta^a_{l,t} = 0$ for all $l \in J_t \backslash \{j, S_k\}$. For simplicity, we assume that agents exclusively issue contract j in equilibrium. Ensuring that $\hat{J}_t$ includes the optimal contract guarantees that constrained agents do not deviate.

In summary, we have verified that in a FREE, in which a finite set of contracts is traded in equilibrium, agents' optimization and market clearing conditions in a collateral equilibrium are satisfied.

IV. Results

The procedure of finding a collateral equilibrium stops when the set of actively traded contracts $\hat{J}_t$ contains only the riskless contract that promises q^D_{t+1} and the risky contract that promises q^U_{t+1}. Appendix G provides a detailed description of the solution accuracy. Let $j_{A,t}$ and $j_{B,t}$ denote the riskless contract and the risky contract respectively. Their corresponding prices are denoted by $\pi_{A,t}$ and $\pi_{B,t}$. The quantities of these contracts held by an age-a agent are denoted by $\theta^a_{A,t}$ and $\theta^a_{B,t}$. In this equilibrium, $\theta^a_{j,t} = 0$ for all $a \in \boldsymbol{A}$, $j \in J_t \backslash \hat{J}_t$. I simulate the economy for a total of $T = 500,000$ periods. All subsequent analyses are based on this simulation[11].

A. Life Cycle Implications

Figure 6 presents the mean life cycle profiles of portfolio holdings in the riskless contract j_A (θ_A), the risky contract j_B (θ_B), housing assets, and the amount of housing pledged as collateral. The collateral profile consistently lies below the housing profile, due to the collateral constraints. This figure suggests that, on average, agents within the model transition through four distinct life stages.

First stage: constrained borrowers. — In the initial stage, agents aged between 1 and 11 choose the portfolio holdings where $\theta^a_{A,t} = 0$, and $-\theta^a_{B,t} = h^a_t$, which indicates that they simultaneously accumulate housing h^a_t and pledge all of these housing assets as collateral to borrow through exclusively selling the risky contract $j_{B,t}$. Because their collateral constraint is binding and they have positive collateral value, they are classified as constrained borrowers.

Anticipating a substantial increase in endowments in the $z_{t+1} = U$ state and a smaller rise in the $z_{t+1} = D$ state, these young agents have two consumption-smoothing motives. First, they would like to transfer wealth from the $z_{t+1} = U$ state, in which they expect to be wealthier, to the present by issuing debt contracts secured by their housing. Second, they would like to make small payments in the $z_{t+1} = D$ state while borrowing as much as possible in the current period. Among all contracts in J_t, the risky contract $j_{B,t}$ allows for the most current borrowing and agents can default on this contract and only deliver the housing

[11] The starting state is selected within the ergodic set of the state space, eliminating the need to discard the initial portion of the simulation to allow the economy to reach its stationary equilibrium.

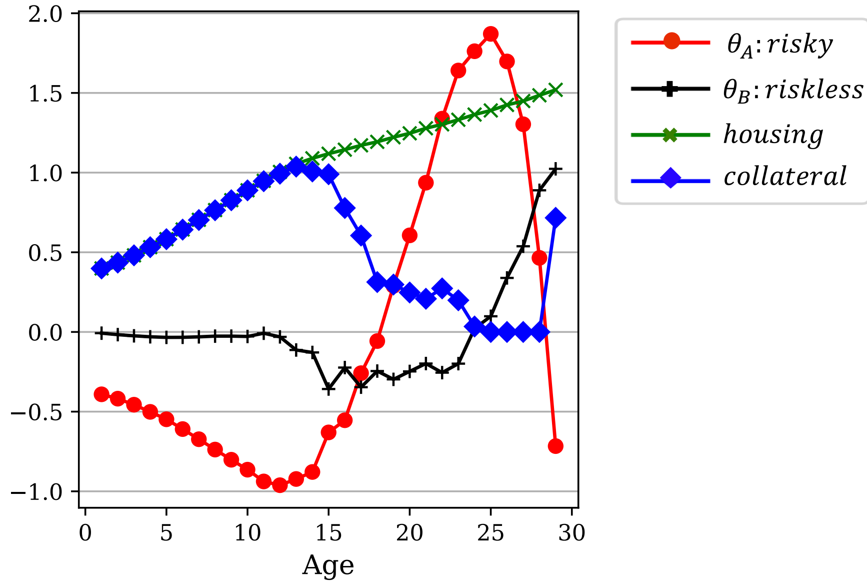

FIGURE 6. AVERAGE LIFE CYCLE PORTFOLIO HOLDINGS

collateral with low value q^D_{t+1} in the D state. Consequently, contract $j_{B,t}$ allows for the greatest degree of consumption smoothing across time and states, these agents will find it most advantageous to exclusively issue this particular contract.

To illustrate that the risky contract is indeed optimal for these constrained young households, Figure 7 examines the liquidity values for age-1 agents. In this figure, there are 500,000 curves, each depicting the liquidity value $LV^{a=1}_{j,t}$ against the promise j across all simulated periods. All curves are color-coded: red for periods when $z_t = U$ and blue for periods when $z_t = D$. Note that all Up state curves are above the Down state ones, implying that the liquidity value for all contracts is always higher in normal states than in crisis states. In Up states, these agents find that the risky contract $j_{B,t}$ provides a strictly higher liquidity value than other contracts whose promises are smaller. Consequently, they use their entire housing collateral to issue this contract, maximizing their current consumption.

The liquidity value in the Down state ($z_t = D$) requires a more cautious analysis. In crisis states, although the liquidity value of the risky contract is equal to that of the riskless contract, the latter provides a smaller amount of borrowing for a given unit of collateral. Therefore, agents continue to find it optimal to issue the risky contract $j_{B,t}$ to satisfy their demand for current consumption without violating their collateral constraints. This choice allows them to maintain the highest possible consumption level during the downturn, even as the Credit Surface steepens.

To understand why the risky contract yields the same liquidity value as the riskless contract in the D state, we must examine the relative endowment growth of the borrowers and lenders. In the D state, both young households and middle-aged lenders expect a substantial endowment increase if the economy transitions to the U state tomorrow. Consequently, in equilibrium, their stochastic discount factors for the U state end up aligning. Therefore, they value the marginal dollar of the state-contingent wealth similarly. However, if the economy remains in the D state tomorrow, young households still anticipate a rise in endowment, whereas middle-aged agents expect a much smaller increase. This makes the middle-aged lenders' SDF significantly higher than that of the young for the future D state, ensuring that liquidity values increase as the promise j moves toward the riskless threshold. Once the promise exceeds q^D_{t+1}, the additional payoff only occurs in the U state, where the young and middle-aged share the same SDF. Therefore, the liquidity value ceases to increase beyond the riskless promise, resulting in the flat profile observed in crisis states.

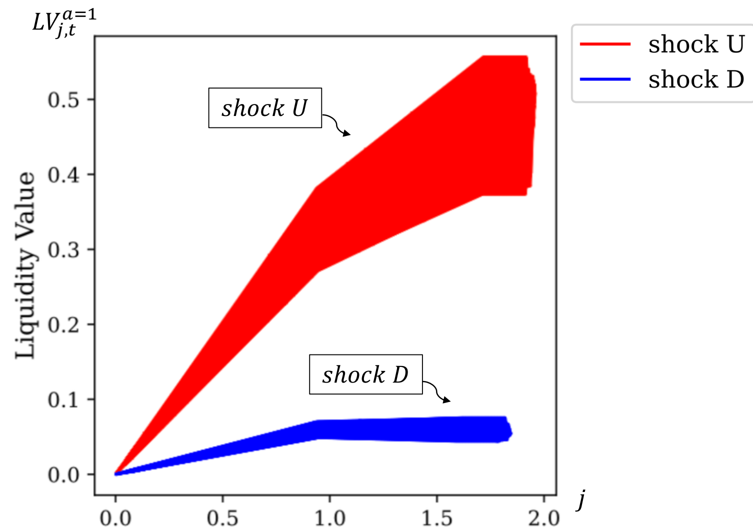

FIGURE 7. LIQUIDITY VALUE FOR $a = 1$ AGENTS

SECOND STAGE: UNCONSTRAINED BORROWERS. — As agents age into the 12 to 18 bracket, they continue to expect higher future endowments and have the motive to transfer future wealth to the present. However, as the average growth rate of their endowment declines compared to that in the initial life stage, their motivation for consumption smoothing declines correspondingly. They start to shift their portfolio holdings away from exclusively issuing the risky contract $j_{B,t}$ to issuing both the riskless contract $j_{A,t}$ and the risky contract $j_{B,t}$. The collateral constraint does not bind throughout this stage. They are indifferent between issuing any contracts in J_t as they are unconstrained, their portfolio holdings in contract $j_{A,t}$ and $j_{B,t}$ are pinned down in equilibrium by the market clearing conditions.

THIRD STAGE: UNCONSTRAINED BORROWERS AND LENDERS. — As agents progress into the 19 to 23 age bracket, they anticipate their future endowments to be on a downward trajectory. This expected change is reflected in Figure 6, where the line of the use of collateral exhibits a mild increase at age 22. This uptick suggests that agents, upon experiencing the initial decline in their endowment growth at age 22, use greater leverage in accumulating housing assets than when they were age-21. During this stage, while utilizing the housing collateral to borrow with contract $j_{A,t}$, agents start to transfer current wealth to the future by investing in the risky contract $j_{B,t}$.

LAST STAGE: LENDERS. — In the final life stage, agents, driven by a strong saving motive and a motive to smooth consumption across states, stop borrowing and begin to diversify their investments. They acquire housing assets with zero leverage, and diversify their saving by lending to younger agents through buying both the risky contract $j_{B,t}$ and the riskless contract $j_{A,t}$. Rearranging agents' Euler equations for contract $j_{A,t}$ and $j_{B,t}$, we obtain the following condition: the excess real return from holding the risky contract over the riskless contract is zero after adjusting for agents' risk preferences.

$$E_t\left[M^a_{t,t+1}\left(\frac{min\{q^U_{t+1}, q_{t+1}\}}{\pi_{B,t}} - \frac{q^D_{t+1}}{\pi_{A,t}}\right)\right] = 0. \tag{33}$$

This condition, together with market clearing conditions, determines agents' portfolio holdings.

LIFE CYCLE PROFILE OF LEVERAGE. — Figure 8 presents the mean DLTV for all age groups throughout the simulated periods. On average, DLTV progressively decreases with age and eventually reaches zero, with the exception of an uptick at age 22, where agents begin to face declines in their endowments.

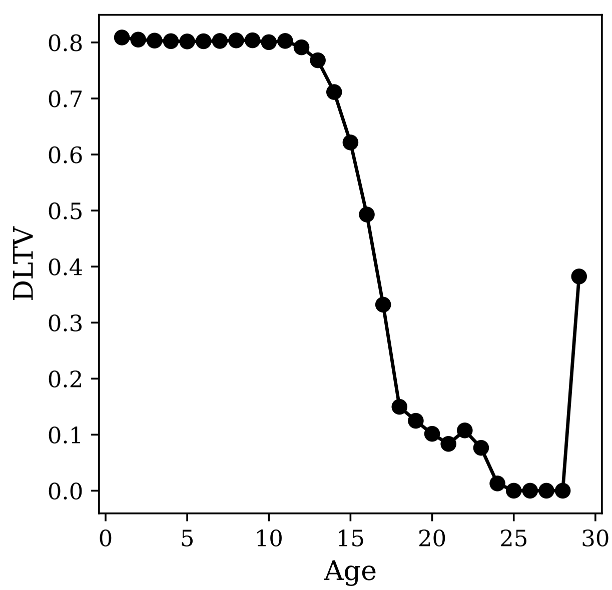

FIGURE 8. AVERAGE DLTV

B. Leverage Cycle Implications

This model features infrequent disaster events. Within this framework, lenders collectively provide pricing schedules for leverage through the Credit Surface, which they adjust endogenously in response to fundamentals. The model generates leverage cycles characterized by the following dynamics: household leverage is positively correlated with housing prices, and leverage amplifies the impact of fundamental shocks on housing prices; mortgage rates and spreads move in the opposite direction, remaining low during normal times while increasing drastically in downturns.

THE CREDIT SURFACE AND LEVERAGE. — Figure 9 illustrates the Credit Surfaces across the 500,000 simulated periods under the baseline parameterization. Table 3 presents the average prices and leverage in states U and D, as well as the differences (Δ), which represent either the absolute change ($D-U$) or the percentage drop ($(D-U)/U$) in the average value of each variable in state D compared to state U. I focus on state-conditional means rather than unconditional correlations because the economy's dynamics are driven by discrete regime shifts between normal times and disaster events. The average Credit Surfaces for states U and D are in darker shades of red and blue, respectively. This figure reveals distinct patterns in the Credit Surface based on the aggregate state. Credit Surfaces associated with D states consistently sit above those for U states, with a tendency to start rising at a lower level of LTV and at a faster speed, becoming vertical at comparatively lower LTV levels. When the economy is in a downturn, the Credit Surface rises and becomes steeper as lenders tighten credit.

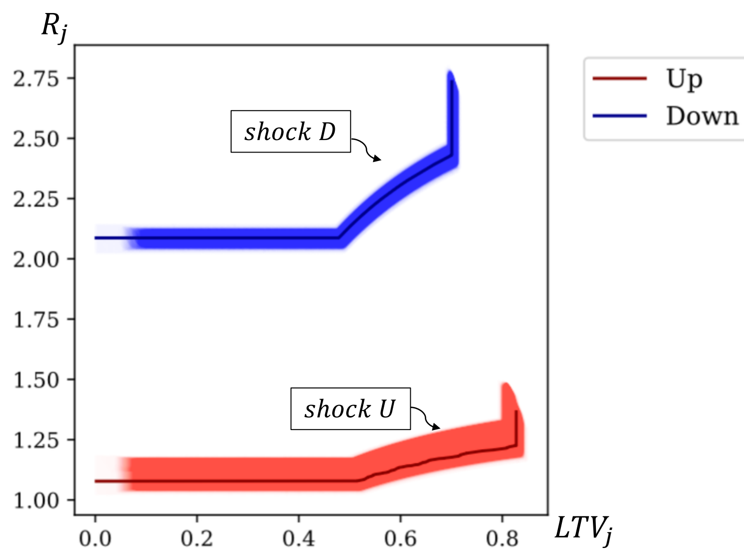

FIGURE 9. CREDIT SURFACES

The leverage of the risky contract differs markedly between states, while the leverage of the riskless contract experiences mild changes. On average, households can leverage as high as 51.4% at a riskless interest rate with the riskless contract $j_{A,t}$ during normal times; in downturns, it only drops mildly to 49.9%. In contrast, the LTV associated with the risky contract $j_{B,t}$, which endogenously sets the leverage cap, is far more volatile, falling from 82.8% in normal times to 72% in downturns.

To understand these distinct patterns, consider the equilibrium pricing equations for both contracts. Lenders

TABLE 3—SUMMARY STATISTICS OF THE COLLATERAL EQUILIBRIUM

Variable	Meaning	U	D	Δ
q	Housing Price	1.70	0.94	-45%
π_A	Riskless Contract Price	0.87	0.47	-46%
π_B	Risky Contract Price	1.40	0.68	-52%
R_A	Riskless Interest Rate	1.08	2.00	+86%
R_B	Risky Interest Rate	1.22	2.36	+94%
$R_B - R_A$	Mortgage Spread	0.1453	0.3621	+21.7 *pp*
LTV_A	Riskless LTV	51.4%	49.9%	-1.5 *pp*
LTV_B	Risky LTV	82.8%	72.0%	-10.7 *pp*
D	Aggregate Debt	19.47	9.12	-53.2%
$DLTV^h$	Diluted aggregate Leverage	38.3%	32.4%	-5.9*pp*

Note: *pp* stands for percentage points. Column U and column D show the mean of each variable conditional on the exogenous state, while column δ presents their differences across states.

price both contracts according to

$$\begin{aligned}
\pi_{A,t} &= E_t[M^a_{t,t+1} q^D_{t+1}] \\
&= E_t[M^a_{t,t+1}] q^D_{t+1} + \underbrace{Cov_t(M^a_{t,t+1}, q^D_{t+1})}_{=0}, \\
\pi_{B,t} &= E_t[M^a_{t,t+1} q_{t+1}] \\
&= E_t[M^a_{t,t+1}] E_t[q_{t+1}] + \underbrace{Cov_t(M^a_{t,t+1}, q_{t+1})}_{<0}.
\end{aligned}$$

The riskless contract is shielded from the covariance penalty, as its payoff q^D_{t+1} is constant across future states. However, the price of the risky contract is penalized by a negative covariance term, as it provides higher payments in states where lenders' marginal utility is low. As aforementioned, when today is a downturn ($z_t = D$), middle-aged lenders face a much smaller increase in their future income than young borrowers if the economy stays in a downturn tomorrow ($z_{t+1} = D$). Consequently, lenders are much more eager to receive a dollar in that future D state than in the U state.

Because the riskless contract $j_{A,t}$ guarantees a full payment in that D state, lenders' high eagerness to be paid keeps its price high. Even as current housing prices q_t fall, this high demand for D-state payments prevents the LTV of the riskless contract from collapsing. Conversely, while the risky contract also pays out the collateral value q^D_{t+1} in a downturn tomorrow, the extra promise it offers over the riskless contract only pays out if the economy transitions to a U state tomorrow. Since lenders are not nearly as eager to be paid in the U state, that additional payment is heavily discounted. Consequently, the price of the risky contract collapses, resulting in a 10.8-percentage-point decline in its LTV.

HOUSING PRICE COLLAPSE. — In this model, housing plays three critical roles, each influencing housing prices. First, housing provides immediate utility. As outlined in (22), the initial component of housing prices is rent: agents must pay a positive amount upfront to use the property. Second, as housing assets are perfectly durable, they serve as a means of savings. Their value today is tied to the present value of future housing prices, as reflected in the second component of housing prices. The third component represents the collateral value of housing, which is derived specifically from its role as collateral.

Housing prices decline sharply when the economy experiences negative endowment shocks. The average housing price in state D is 45% lower than in state U. This decline stems from the simultaneous fall in all three components of housing prices, driven by two interconnected channels, each affecting a distinct type of marginal buyers: the reduced wealth of middle-aged marginal buyers and the diminished leverage capacity of constrained young borrowers.

The wealth channel (middle-aged lenders). Households face lower endowments in downturns, and middle-aged lenders are hit particularly hard because they save significantly in housing and debt contracts, both of which suffer substantial losses. When young borrowers default, instead of receiving the repayment of $j = q^U_{t+1}$, lenders can only liquidate the collateral at a lower housing price, q^D_{t+1}. Additionally, their housing wealth declines as housing prices fall during downturns. With diminished wealth, all households, especially lenders, are unable to support high rent payments for housing as they could in normal states. Households experience a sharp increase in the marginal utility of consumption, which leads to a heavier discounting of the future and reduces the present value of future housing prices.

The leverage channel (young borrowers). Most importantly, leverage is significantly reduced because the capacity of housing assets to facilitate loans declines sharply. The average prices of riskless and risky contracts, π_A and π_B, are 46% and 52% lower in state D compared to state U, respectively, exceeding the magnitude of the decline in housing prices (45%). For constrained borrowers, collateral value is a major component of housing prices, while it is zero for unconstrained borrowers and lenders. As a result, this channel impacts constrained borrowers the hardest.

To examine how the collateral value for each constrained borrower responds to a one-time D shock, I simulate the economy 100,000 times over 20 periods. In each simulation, the economy starts in the U state at time 0, experiences a D shock at time 1, and returns to the U state thereafter. For each time period, I calculate the deviation of each variable relative to its value at time 0 and then average these deviations across all simulations. Figure 10 reports the average percentage deviations in collateral value for age groups 1 to 11. On impact, the collateral value for these constrained borrowers declines sharply, around 80%. As the third component of housing prices, this decline feeds back to further reductions in housing prices.

Despite its substantial decline, housing remains less volatile than the debt contracts. While housing and risky debt share the same discounted continuation value, housing uniquely provides immediate service-flow utility. Because this rent component is a current flow variable, it is not subject to the risk-aversion penalty applied to future state-contingent claims. Although it declines during downturns as consumption falls, the magnitude of this drop is significantly smaller than that of the other components. Consequently, this utility flow acts as a cushion for housing prices, ensuring they fall less than contract prices.

C. A Benchmark: The Bond Economy

To assess the impact of endogenously determined leverage on housing price volatility, I analyze a simple bond economy in which this mechanism is absent. The bond economy differs from the baseline collateral economy in three fundamental ways: the only financial asset available is a riskless bond in zero net supply; agents are assumed to never default; and the LTV ratio is exogenously capped at ϕ.

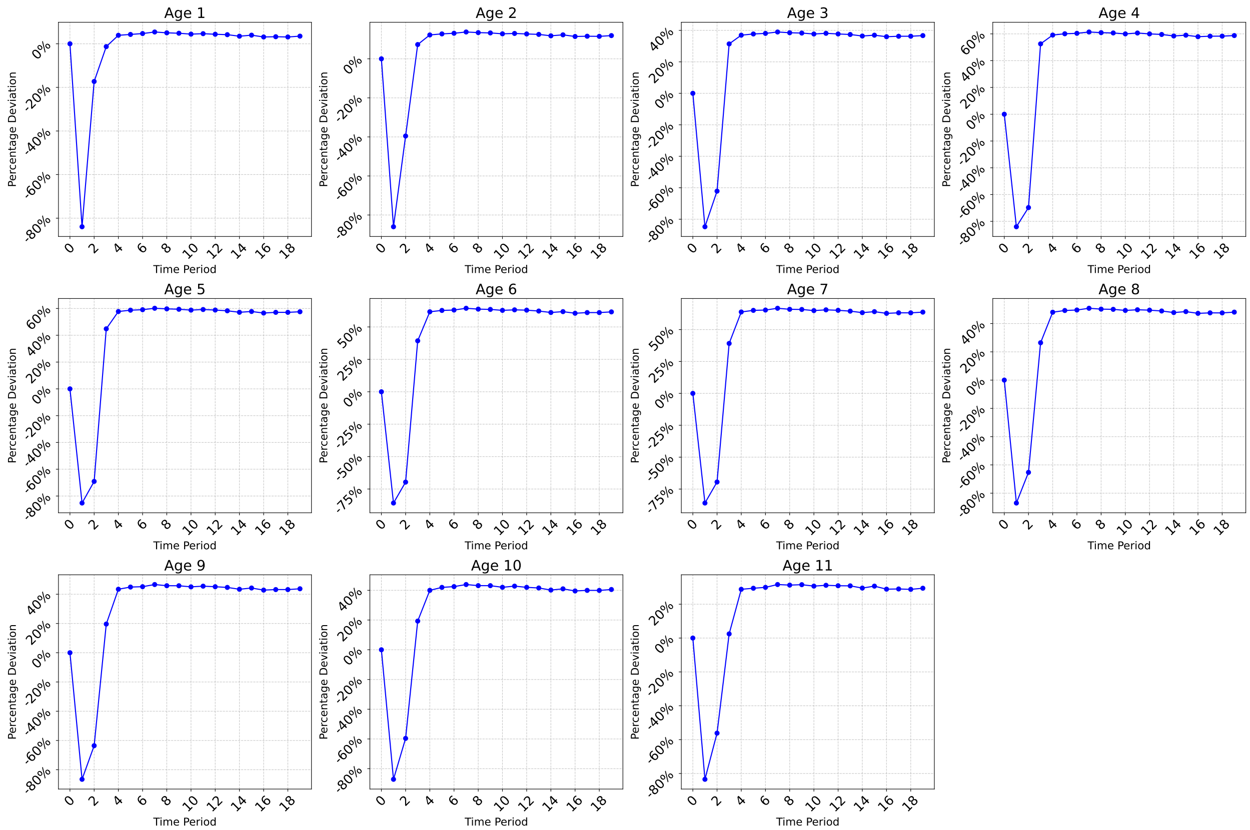

FIGURE 10. PERCENTAGE DEVIATIONS IN COLLATERAL VALUE FOR CONSTRAINED BORROWERS UNDER A D SHOCK

Note: For each simulation, I select the initial state vector from the states generated during the 500,000-period simulation where the exogenous state is U. Therefore, each simulation begins with an initial state within the ergodic set of the state space.

ENVIRONMENT. — Let $(b_t^a)_{a\in\boldsymbol{A}}$ denote the bond holdings of agents, and p_t the price of the bond. Agents, taking prices as given, maximize their life-time utility (1) subject to the budget constraint:

$$c_t^a + q_t h_t^a + p_t b_t^a \leqslant e_t^a + q_t h_{t-1}^{a-1} + b_{t-1}^{a-1},$$

and a borrowing constraint, as is standard in the macroeconomic housing literature:

$$-p_t b_t^a \leqslant \phi q_t h_t^a.$$

LTV of an agent is given by:

$$LTV_t^a = \frac{max\{-p_t b_t^a, 0\}}{q_t h_t^a}.$$

In the bond economy, interest rates no longer increase in leverage as default risk is eliminated. All loans within the feasible range of leverage are subject to a uniform interest rate, which is the inverse of the bond price p_t:

$$R_t = \frac{1}{p_t}.$$

The aggregate amount of debt in the bond economy is given by the product of the bond price and the aggregate quantity of bonds shorted across agents:

$$p_t \sum_{a \in \boldsymbol{A}} max\{-b_t^a, 0\}.$$

Since bond issuance is not directly tied to the quantity of housing assets, I do not discuss the LTV of housing in this economy. Instead, I define the diluted LTV of housing as the ratio of aggregate debt in the economy to the aggregate value of housing:

$$DLTV_t^h = \frac{p_t \sum_{a \in \boldsymbol{A}} max\{-b_t^a, 0\}}{q_t H}.$$

While the bond economy can generate housing price fluctuations, it lacks the two primary amplification channels present in the collateral economy. First, the covariance pricing channel is absent. In this environment, lenders price a single riskless bond. Since the bond payoff is identical across states, its pricing does not involve a covariance term. In contrast, in the collateral economy, risky contracts are heavily discounted in downturns because their payoffs are negatively correlated with lenders' marginal utility. This repricing tightens credit and leads to endogenous deleveraging.

Second, the financial wealth of lenders is less volatile. In the collateral economy, the downturn triggers defaults on risky contracts, which erodes lenders' wealth and reduces credit supply. In the bond economy, however, lenders' wealth is shielded from default risk, and the feedback from lenders' balance sheets to credit supply is largely absent.

NUMERICAL RESULTS AND COMPARISONS. — Using the average upper limit of LTV in the normal state of the collateral economy as a reference point, I set $\phi = 83\%$ and compute the equilibrium. I then simulate the bond economy for 500,000 periods. Table 4 presents an overview of key variables across the two economies.

Two quantitative facts stand out. First, housing prices are more volatile in the collateral economy compared to the bond economy, which can be seen by comparing the coefficient of variation of the housing price across these two economies. In addition, the average housing price in state D is 45% below the U mean in the collateral economy, but only 32% below in the bond economy. Second, aggregate debt and diluted housing leverage behave very differently: aggregate debt falls by 53% in the collateral economy (compared to only 22% in the bond economy), and $DLTV^h$ falls by 5.9 percentage points in the collateral economy but rises by 4.5 percentage points in the bond economy.

Admittedly, with a plausible calibration, such as an exogenous decline in the LTV cap ϕ during downturns, the bond economy has the potential to match the observed movements in housing prices and leverage. The purpose of this comparison is to illustrate the mechanism needed to generate leverage cycles endogenously.

Figure 11 compares the unconditional deviations of these key variables caused by a one-time D shock at time 1 across the two economies. The results echo the previous findings: housing prices and aggregate

TABLE 4—COMPARISON OF COLLATERAL ECONOMY AND BOND ECONOMY

	Collateral Economy			Bond Economy		
	U	D	Δ	U	D	Δ
Housing Price	1.7	0.94	-45%	1.66	1.13	-32%
Aggregate debt	19.5	9.1	-53%	14.6	11.4	-22%
$DLTV^h$	38.3%	32.4%	-5.9 pp	29.3%	33.8%	4.5 pp
Volatility		17.5%			13%	

Note: Volatility refers to the volatility of housing prices, which is measured by the coefficient of variation over the simulation. 16.2% indicates that housing prices deviate by 16.2% from the mean on average.

debt decline more sharply and recover more slowly in the collateral economy than in the bond economy. Consistent with the previous findings, leverage declines in the collateral economy but increases in the bond economy. On impact, housing leverage drops by 6 percentage points in the collateral economy but rises by nearly 4 percentage points in the bond benchmark.

These differences are attributed to the composition of debt and its impact on lenders' balance sheets. In the collateral economy, over 80% of aggregate debt is issued through the risky contract $j_{B,t}$, causing lenders to suffer large default losses during downturns. Figure 12 illustrates this wealth channel, where lenders in the collateral economy experience a much greater decline in financial wealth.

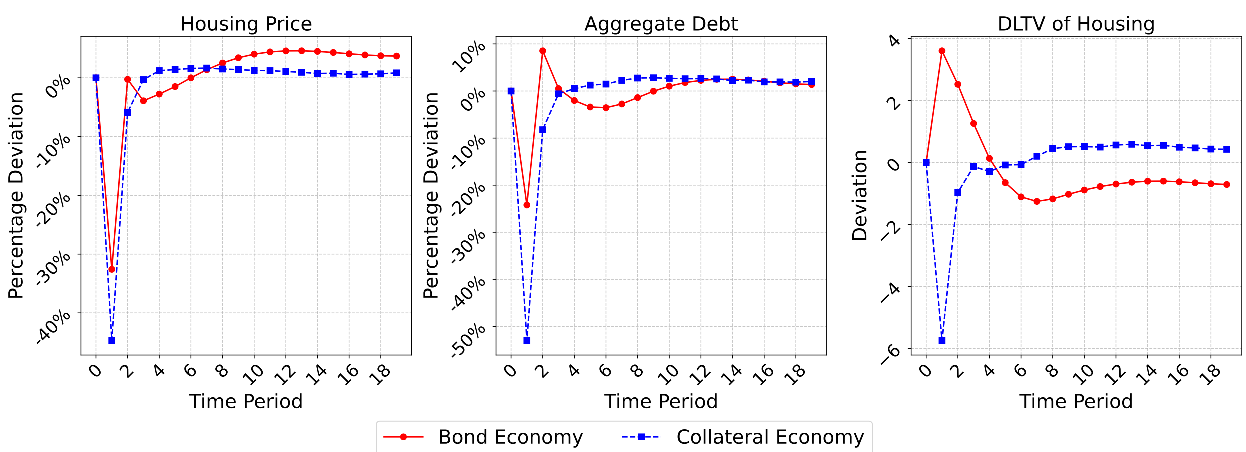

FIGURE 11. DEVIATIONS IN HOUSING PRICE, AGGREGATE DEBT AND LEVERAGE UNDER A D SHOCK: COLLATERAL VS BOND ECONOMIES

Note: Deviations in housing prices and credit are expressed as percentages, while deviations in leverage ($DLTV^h$) are in percentage points.

V. Conclusion

This paper develops a quantitative model to rationalize the comovements of housing prices, leverage, and mortgage spreads in the U.S. housing market. The model demonstrates that the LTV limits and the risk-based

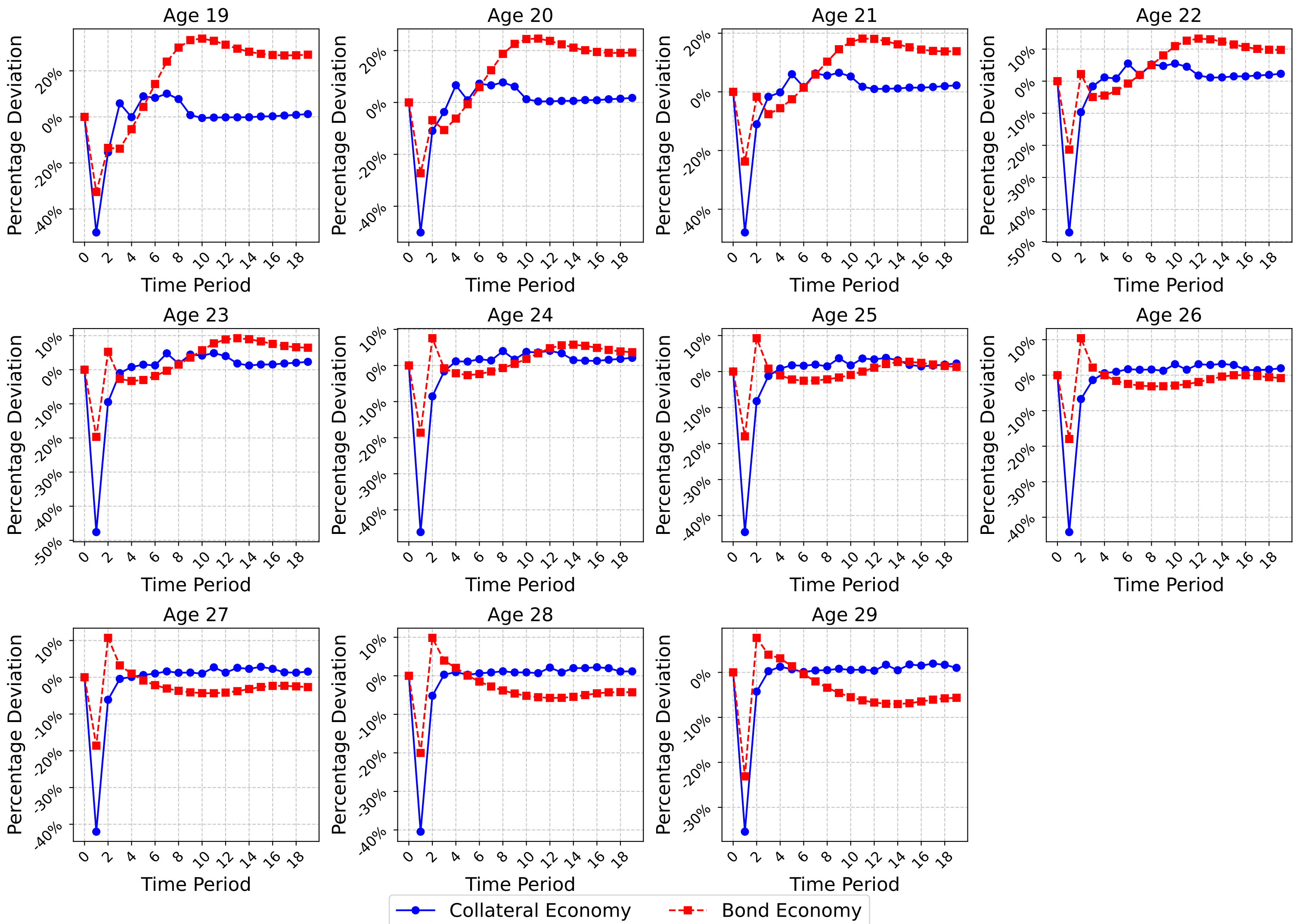

FIGURE 12. PERCENTAGE DEVIATIONS IN FINANCIAL WEALTH UNDER A D SHOCK: COLLATERAL VS BOND ECONOMIES

Note: Financial wealth is the equity after clearing debts: $q_t h_{t-1}^{a-1} + \int_{R_+} \theta_{j,t-1}^{a-1} min\{j, q_t\} dj$ for the collateral economy, and $q_t h_{t-1}^{a-1} + b_{t-1}^{a-1}$ for the bond economy.

pricing of debt, the Credit Surface, are inherently dynamic, adjusting endogenously in response to economic fundamentals. This framework challenges the conventional assumption of exogenously fixed leverage limits and provides a quantitative rationale for how housing finance conditions fluctuate across aggregate states.

The study highlights how the life cycle of households interacts with financial markets to create these dynamics. Young households, anticipating future income growth, use high leverage to accumulate housing. Middle-aged and older cohorts lend to the young by holding portfolios of risky and riskless debt contracts. When a negative aggregate shock hits, young borrowers default on risky contracts, eroding the financial wealth of lenders and depressing housing prices. Simultaneously, lenders heavily discount contracts with default risk, leading to higher returns on lending and causing the Credit Surface to rise and steepen. This reduction in leveraging capacity further depresses housing prices, creating a powerful feedback loop that amplifies the initial shock.

Crucially, the framework shifts the focus from borrower-side constraints to lender-side pricing of debt. With risk-based pricing over a continuum of contracts, the model informs how lenders' stochastic discounting determines the level and the shape of the Credit Surface. These findings suggest that macroprudential policies aimed at stabilizing the housing market should be evaluated through their impact on the *entire* Credit Surface, rather than solely through the riskless segment.

The risky segment of the surface consists of mortgage loans with default risk, which are the most sensitive to lenders' discounting during downturns. In the model, while the riskless segment remains relatively stable, the risky segment experiences a sharp contraction in credit supply as lenders' marginal utility rises and their wealth is impaired. Policies that maintain stability only at low leverage levels may fail to prevent the risky segment from becoming excessively steep and the subsequent crash of housing prices. Simply setting a static LTV cap is insufficient, as it does not address the underlying shifts in lenders' risk-bearing capacity that drive the endogenous collapse of credit supply. A policy framework informed by this model would instead target the Credit Surface to mitigate the leverage-driven volatility inherent in housing markets.

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APPENDIX A: PROOF OF PROPOSITION 1

PROOF:

Let y_t^z denote the variable y in state z_t. Since households have strictly positive endowments, and the period utility functions satisfy $\lim_{c\to 0} Du_c^a = \infty$ and $\lim_{h\to 0} Du_h^a = \infty$ for $a \in \hat{\boldsymbol{A}}$ and $\lim_{c\to 0} Du_c^A = \infty$, it follows that $c_t^a > 0$ and $Du_c^a > 0$ for all $a \in \boldsymbol{A}$, $h_t^a > 0$ and $Du_h^a > 0$ for all $a \in \hat{\boldsymbol{A}}$, and that q_t is strictly positive.

Consider in equilibrium, an age-a agent makes a net purchase of contract j, it must be that the price of the contract is equal to the present value of its delivery. The delivery is discounted by the agent's intertemporal marginal rate of substitution of consumption between time t and $t+1$:

$$\begin{aligned}\pi_{j,t} &= E_t\left[\beta \frac{Du_c^{a+1}(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c^a(c_t^a, h_t^a)} min\{j, q_{t+1}\}\right] \\ &= \sum_{z_{t+1}\in \boldsymbol{Z}} \beta \gamma_{z_t, z_{t+1}} \frac{Du_c^{a+1}(c_{t+1}^{a+1,z_{t+1}}, h_{t+1}^{a+1,z_{t+1}})}{Du_c^a(c_t^a, h_t^a)} min\{j, q_{t+1}^{z_{t+1}}\}. \end{aligned} \tag{A1}$$

Let $M_{t,t+1}^{a,z_{t+1}}$ represent the state-dependent intertemporal marginal utility,

$$M_{t,t+1}^{a,z_{t+1}} \equiv \beta \gamma_{z_t, z_{t+1}} \frac{Du_c^{a+1}(c_{t+1}^{a+1,z_{t+1}}, h_{t+1}^{a+1,z_{t+1}})}{Du_c^a(c_t^a, h_t^a)},$$

$M_{t,t+1}^{a,z_{t+1}}$ is strictly positive for all $z_{t+1} \in \boldsymbol{Z}$. At equilibrium prices and quantities, the second line of (A1) shows that $\pi_{j,t}$ is a continuous, piecewise linear function of j for $j \in \mathbb{R}_+$. The slope is $M_{t,t+1}^{a,U} + M_{t,t+1}^{a,D}$ for $j \in [0, q_{t+1}^D]$ and $M_{t,t+1}^{a,U}$ for $j \in (q_{t+1}^D, q_{t+1}^U]$. Therefore, $\pi_{j,t}$ is nonnegative and strictly increasing in j over $[0, q_{t+1}^U]$.

For contracts with the promise $j \in (q_{t+1}^U, \infty)$, their delivery is q_{t+1}^D in $z_{t+1} = D$ and q_{t+1}^U in $z_{t+1} = U$, which equals the delivery of the contract with $j = q_{t+1}^U$. Hence, $\pi_{j,t} = \pi_{q_{t+1}^U, t}$ for $j \in (q_{t+1}^U, \infty)$.

Since q_t is strictly positive, it follows that $LTV_{j,t}$ is also a nonnegative, piecewise strictly increasing function of j in $[0, q_{t+1}^U]$, with a slope proportional to the slope of $\pi_{j,t}$ with respect to j, scaled by the inverse of q_t. For $j \in (q_{t+1}^U, \infty)$, $LTV_{j,t}$ remains constant at $LTV_{q_{t+1}^U, t}$.

APPENDIX B: PROOF OF PROPOSITION 2

PROOF:

Consider three cases: $j \in [0, q_{t+1}^D]$, $j \in (q_{t+1}^U, \infty)$, and $j \in (q_{t+1}^D, q_{t+1}^U]$.

1. Case 1: $j \in [0, q_{t+1}^D]$

According to (A1), in equilibrium, the interest rate $R_{j,t}$ is constant over this interval, and equal to the inverse of the expected intertemporal marginal utility of consumption of unconstrained agents:

$$R_{j,t} = \frac{j}{\pi_{j,t}} = \frac{1}{\sum_{z_{t+1}\in Z} M_{t,t+1}^{a,z_{t+1}}}.$$

2. Case 2: $j \in (q^U_{t+1}, \infty)$

As j goes to infinity, $LTV_{j,t}$ remains constant at $LTV_{q^U_{t+1},t}$, while $R_{j,t}$ approaches infinity. Thus, the Credit Surface becomes a vertical line for $j > q^U_{t+1}$.

3. Case 3: $j \in (q^D_{t+1}, q^U_{t+1}]$

For this interval, $LTV_{j,t}$ is given by:

$$LTV_{j,t} = \frac{M^{a,D}_{t,t+1} q^D_{t+1} + M^{a,U}_{t,t+1} j}{q_t},$$

and the interest rate is:

$$R_{j,t} = \frac{j}{M^{a,D}_{t,t+1} q^D_{t+1} + M^{a,U}_{t,t+1} j}.$$

Using the chain rule, the derivative of $R_{j,t}$ with respect to $\pi_{j,t}$ is:

$$\begin{aligned} \frac{dR_{j,t}}{d\pi_{j,t}} &= \frac{dR_{j,t}}{dj} \frac{dj}{d\pi_{j,t}} \\ &= \frac{\pi_{j,t} - j\frac{d\pi_{j,t}}{dj}}{(\pi_{j,t})^2} \cdot \frac{1}{\frac{d\pi_{j,t}}{dj}} \\ &= \frac{M^{a,D}_{t,t+1} q^D_{t+1}}{(\pi_{j,t})^2 M^{a,U}_{t,t+1}} > 0. \end{aligned}$$

Since $q_t > 0$, it follows that $\frac{dR_{j,t}}{dLTV_{j,t}} > 0$. Furthermore, observe that $\frac{dR_{j,t}}{d\pi_{j,t}}$ is decreasing in $\pi_{j,t}$, which implies that $\frac{dR_{j,t}}{dLTV_{j,t}}$ is also decreasing in $LTV_{j,t}$. Hence, $R_{j,t}$ is increasing and concave in $LTV_{j,t}$ for $j \in (q^D_{t+1}, q^U_{t+1}]$.

APPENDIX C: PROOF OF PROPOSITION 3

PROOF:

Suppose that, in equilibrium, there exists a contract $k \in J_t$ such that $k > q^U_{t+1} > q^D_{t+1}$ and $LTV_{k,t} \geqslant 100\%$. By definition, this implies that $\pi_{k,t} \geqslant q_t$. At equilibrium, the following conditions must hold for agents whose collateral constraint is not binding:

$$\begin{aligned} \pi_{k,t} &= E_t\left[\beta \frac{Du^{a+1}_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du^a_c(c^a_t, h^a_t)} q_{t+1}\right], \\ q_t &= \frac{Du^a_h(c^a_t, h^a_t)}{Du^a_c(c^a_t, h^a_t)} + E_t\left[\beta \frac{Du^{a+1}_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du^a_c(c^a_t, h^a_t)} q_{t+1}\right]. \end{aligned}$$

The second equation states that the housing price q_t equals the sum of the immediate utility from housing services and the expected present value of the future housing price. Taking the difference between q_t and $\pi_{k,t}$, we get the following equation, which shows that the down payment associated with this contract equals the utility derived from a single period of housing consumption:

$$q_t - \pi_{k,t} = \frac{Du_h^a(c_t^a, h_t^a)}{Du_c^a(c_t^a, h_t^a)}.$$

Since $\frac{Du_h^a(c_t^a, h_t^a)}{Du_c^a(c_t^a, h_t^a)}$ must be strictly greater than zero, it follows that $q_t > \pi_{k,t}$. This result contradicts the initial assumption $\pi_{k,t} \geqslant q_t$, therefore, $LTV_{k,t} < 100\%$ for all contracts such that $k > q_{t+1}^U > q_{t+1}^D$. Since the prices of all other contracts in J_t are less than or equal to $\pi_{k,t}$ by Proposition 1, it follows that no contract in J_t can have a price greater than or equal to q_t. Hence, the upper limit of $LTV_{j,t}$ is strictly less than 100%[12].

Appendix D: Sensitivity Analysis

This appendix presents a sensitivity analysis of the model with respect to the coefficient of relative risk aversion (ρ). Table D1 shows that higher ρ monotonically amplifies the magnitude of movements in housing prices and credit conditions. The underlying mechanism is that a higher ρ increases the sensitivity of lenders' marginal utility to aggregate shocks, leading them to discount payoffs that are negatively correlated with their marginal utility more aggressively. Note that in the special case where $\rho = 1$ (log utility), marginal utility varies less across aggregate states, and consequently equilibrium adjustments in asset prices and leverage remain comparatively small. This shift in state-contingent discounting is reflected in contract prices, LTV, and aggregate debt. As ρ increases, the magnitude of the housing price collapse also increases. For instance, the price decline in state D deepens from 41.6% to 46.5% as ρ rises from 5.0 to 6.0.

In terms of contract pricing, prices in the downturn state decline as ρ increases, with risky debt exhibiting greater sensitivity. As a result, the Credit Surface steepens as ρ rises. These price adjustments lead to tighter borrowing conditions, particularly for the risky contract. The contraction in risky leverage from state U to D grows with ρ, with LTV_B falling by 9.1 pp at $\rho = 5.0$ and by 11.8 pp at $\rho = 6.0$. In contrast, the LTV of the riskless contract remains relatively stable across all specifications.

At the aggregate level, both aggregate debt and housing prices decline more sharply as ρ increases. Debt contraction worsens from 49.5% at $\rho = 5.0$ to 56.2% at $\rho = 6.0$, while the decline in diluted leverage expands from 5.2 pp to 6.9 pp. These results confirm that lenders' risk preferences govern the magnitude of the resulting volatility.

Appendix E: Algorithm for Approximating a FREE

1) Set the episode counter $s = 0$. Initialize a neural network $\mathcal{N}^{(s)}$ with three hidden layers. Each of the hidden layer contains 512 nodes.

2) Solve for the steady state equilibrium using the endowments for $z = U$, and use the resulting equilibrium variables to construct the initial state vector $x_0{}^{(s)}$. The state vector lies in the space $\boldsymbol{Z} \times [0, H]^A \times ([-H, H]^{N_j})^A \times [0, 1]^2$.

3) Simulate the economy using $\mathcal{N}^{(s)}$ and state vector $x_0{}^{(s)}$ over 10,000 periods. Construct a training

[12] Nilayamgode (2023) also demonstrates that in a two-period model, in which non-financial assets serve as collateral and households holding housing assets maintain positive levels of consumption, the LTV ratio can never reach 100%.

TABLE D1—SENSITIVITY ANALYSIS: IMPACT OF RISK AVERSION (ρ)

Variable	$\rho = 1$ (Log)			$\rho = 5.0$			$\rho = 5.5$ (Baseline)			$\rho = 6.0$		
	U	D	Δ	U	D	Δ	U	D	Δ	U	D	Δ
q (Housing Price)	2.64	2.32	-11.9%	1.72	1.00	-41.6%	1.70	0.94	-44.8%	1.68	0.90	-46.5%
π_A (Riskless Price)	2.10	1.88	-10.9%	0.92	0.53	-42.7%	0.87	0.47	-46.4%	0.84	0.44	-48.1%
π_B (Risky Price)	2.34	2.06	-12.1%	1.43	0.74	-48.0%	1.40	0.68	-51.9%	1.39	0.64	-54.2%
R_A (Riskless IR)	1.10	1.23	+11.6%	1.09	1.89	+73.8%	1.08	2.00	+86.2%	1.06	2.06	+93.8%
R_B (Risky IR)	1.13	1.27	+12.8%	1.22	2.19	+80.6%	1.22	2.36	+93.7%	1.22	2.47	+101.6%
$R_B - R_A$ (Spread)	0.0210	0.0367	+1.57 *pp*	0.127	0.303	+17.6 *pp*	0.145	0.362	+21.7 *pp*	0.160	0.405	+24.5 *pp*
LTV_A (Riskless LTV)	79.8%	80.8%	+0.97 *pp*	53.8%	52.7%	-1.1 *pp*	51.4%	49.9%	-1.5 *pp*	50.2%	48.7%	-1.5 *pp*
LTV_B (Risky LTV)	89.0%	88.8%	-0.20 *pp*	83.0%	73.9%	-9.1 *pp*	82.8%	72.0%	-10.8 *pp*	82.6%	70.8%	-11.8 *pp*
D (Aggregate Debt)	41.84	34.14	-18.4%	19.73	9.96	-49.5%	19.47	9.12	-53.2%	19.37	8.49	-56.2%
$DLTV^h$ (Diluted LTV)	52.9%	49.0%	-3.88 *pp*	38.3%	33.1%	-5.2 *pp*	38.3%	32.4%	-5.9 *pp*	38.5%	31.5%	-6.9 *pp*

dataset $\mathscr{D}_{\text{train}}^{(s)}$ using the 100,000-period simulation. Each simulation counts as one episode within the training sequence.

4) Calculate the loss function, which is the mean squared errors across all equilibrium conditions. These conditions include the Euler equations, market clearing conditions, budget constraints, and complementary slackness conditions, evaluated over the training dataset $\mathscr{D}_{\text{train}}^{(s)}$.

5) Implement the Adam optimization algorithm, a type of mini-batch stochastic gradient descent, to update the neural network parameters from $\mathscr{N}^{(s)}$ to $\mathscr{N}^{(s+1)}$. Update the parameters only once per simulation. Increment the episode counter to $s = s+1$. Set the new initial state vector $x_0{}^{(s+1)}$ as the final state vector from the preceding simulation.

6) Repeat steps 3 to 5 until either the episode counter reaches $200,000$ or the neural network converges. If the loss function does not converge after 200,000 episodes, adjust the learning rate, the size of mini-batches, or the number of nodes in each hidden layer, then return to step 1 with the new parameters.

Appendix F: Proof that the Liquidity Value of the Optimal Contract Equals the Collateral Value of Housing

In this appendix, using proof by contradiction, I show that the liquidity value of the optimal contract, $j_t^{a,*}$, equals the collateral value of housing for constrained agents. For this proof, agents with both a binding collateral constraint and zero Lagrange multipliers are categorized as unconstrained agents, which are not within the scope of this analysis.

PROOF:

For constrained agents with optimal contract $j_t^{a,*}$, the Lagrange multiplier associated with the issuance of contract $j_t^{a,*}$ must be strictly positive, and the collateral constraint binds:

$$\mu^a_{j_t^{a,*},t} > 0, \quad -\theta^a_{j_t^{a,*},t} - h^a_t = 0.$$

The multiplier $\mu^a_{j_t^{a,*},t}$ is the sum of the Lagrange multipliers associated with all non-empty subsets S of $\hat{J}_t$ that include $j_t^{a,*}$:

$$\mu^a_{j_t^{a,*},t} = \sum_S \mu_t^{a,S}, \quad \forall S \text{ such that } S \neq \emptyset,\ j_t^{a,*} \in S,\ S \subseteq \hat{J}_t.$$

Suppose $\mu^a_{j_t^{a,*},t} \neq \mu^a_{h,t}$. Since $\mu^a_{h,t}$ is the sum of the Lagrange multipliers of all non-empty subsets of $\hat{J}_t$, it must be that:

$$\mu^a_{j_t^{a,*},t} < \mu^a_{h,t}.$$

It follows that the difference between $\mu^a_{h,t}$ and $\mu^a_{j_t^{a,*},t}$ is strictly positive:

$$\mu^a_{h,t} - \mu^a_{j_t^{a,*},t} = \sum_K \mu_t^{a,K} > 0,$$

where K represents all non-empty subsets of $\hat{J}_t$ that do not contain $j_t^{a,*}$.

The condition that $\sum_{K\subseteq\hat{J}_t: j_t^{a,*}\notin K}\mu_t^{a,K} > 0$ indicates that there exists at least one non-empty subset $K \subseteq \hat{J}_t$, with $j_t^{a,*} \notin K$, such that the collateral constraint binds for the set K. This implies that the agent could issue contracts in K to fully utilize their collateral. However, this contradicts the assumption that $j_t^{a,*}$ is the optimal contract that the constrained agent issues exclusively.

Therefore, it must be the case that:

$$\mu^a_{j_t^{a,*},t} = \mu^a_{h,t},$$

and the liquidity value of this optimal contract is equal to the collateral value.

APPENDIX G: ACCURACY OF NUMERICAL SOLUTIONS

This appendix examines the accuracy of the numerical solutions for the collateral economy and the bond economy based on simulations spanning 500,000 periods. The equilibrium of each economy is characterized by four sets of equations: (1) complementary slackness conditions for collateral constraints or borrowing constraints, (2) Euler equations, (3) budget constraints, and (4) market clearing conditions. In both simulations, the budget constraints are enforced, so there are no approximation errors in this aspect.

Tables G1 and G2 report the errors for the remaining three types of equilibrium conditions for the collateral and bond economies, respectively. The errors are expressed in terms of log_{10}. Following Judd (1992), I define the relative Euler equation errors for each type of condition in the collateral and bond economies as follows.

For the collateral economy, the relative Euler equation errors for housing(h^a_t), the riskless contract($\theta^a_{A,t}$), and the risky contract($\theta^a_{B,t}$) are defined as:

$$e(h^a_t) = |1 - \frac{(Du^a_c)^{-1}(\frac{Du^a_h(c^a_t,h^a_t)+\beta E_t[Du^{a+1}_c(c^{a+1}_{t+1},h^{a+1}_{t+1})q_{t+1}]+\mu^a_{h,t}}{q_t})}{c^a_t}|,$$

$$e(\theta^a_{A,t}) = |1 - \frac{(Du^a_c)^{-1}(\frac{\beta E_t[Du^{a+1}_c(c^{a+1}_{t+1},h^{a+1}_{t+1})q^D_{t+1}]+\mu^a_{A,t}}{q_t})}{c^a_t}|,$$

$$e(\theta^a_{B,t}) = |1 - \frac{(Du^a_c)^{-1}(\frac{\beta E_t[Du^{a+1}_c(c^{a+1}_{t+1},h^{a+1}_{t+1})q_{t+1}]+\mu^a_{B,t}}{q_t})}{c^a_t}|.$$

For the bond economy, the relative Euler equation errors for housing (h^a_t) and bond holdings (b^a_t) are:

$$e(h^a_t) = |1 - \frac{(Du^a_c)^{-1}(\frac{Du^a_h(c^a_t,h^a_t)+\phi\mu^a_t q_t+\beta E_t[Du^{a+1}_c(c^{a+1}_{t+1},h^{a+1}_{t+1})q_{t+1}]}{q_t})}{c^a_t}|,$$

$$e(b^a_t) = |1 - \frac{(Du^a_c)^{-1}(\frac{\mu^a_t p_t+\beta E_t[Du^{a+1}_c(c^{a+1}_{t+1},h^{a+1}_{t+1})]}{p_t})}{c^a_t}|,$$

where μ^a_t is the Lagrange multiplier associated with the borrowing constraint.

The maximum Euler equation error remains below 10^{-2}, indicating that the largest percentage loss in consumption due to approximation errors in prices is less than 1%. Errors in the market clearing conditions for housing and the two contracts are normalized by the aggregate housing stock H, while errors in consumption are normalized by the aggregate endowment.

TABLE G1—SUMMARY STATISTICS OF ERRORS IN EQUILIBRIUM CONDITIONS: COLLATERAL ECONOMY

Equilibrium Conditions	Type	25th	Median	99th	Mean	Max
Collateral Constraints		-6.61	-5.45	-4.24	-4.98	-3.85
Euler Equations	Housing	-4.16	-3.89	-3.13	-3.78	-2.12
	Risky	-4.24	-3.89	-3.01	-3.73	-2.05
	riskless	-4.23	-3.91	-3.09	-3.76	-2.10
Market Clearing Conditions	Housing	-4.86	-4.65	-4.20	-4.60	-3.59
	Risky	-5.19	-4.74	-4.17	-4.68	-3.64
	riskless	-5.40	-5.00	-4.22	-4.84	-3.58
	Consumption	-4.87	-4.60	-4.06	-4.51	-3.52

Note: The 25th and 99th refer to the 25th and 99th quantiles of each error in the simulation data series.

TABLE G2—SUMMARY STATISTICS OF ERRORS IN EQUILIBRIUM CONDITIONS: BOND ECONOMY

Equilibrium Conditions	Type	25th	Median	99th	Mean	Max
Borrowing Constraints		-6.41	-5.68	-3.85	-5.01	-2.86
Euler Equations	Housing	-4.72	-4.34	-3.50	-4.17	-2.04
	Bond	-4.69	-4.29	-3.42	-4.10	-2.05
Market Clearing Conditions	Housing	-5.23	-4.96	-4.25	-4.85	-3.18
	Bond	-4.83	-4.50	-4.09	-4.52	-3.12
	Consumption	-5.55	-5.24	-4.62	-5.2	-3.51